

Space functions and complexity of the word problem in semigroups.

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Abstract

We introduce the space function $s(n)$ of a finitely presented semigroup $S = \langle A \mid R \rangle$. To define $s(n)$ we consider pairs of words w, w' over A of length at most n equal in S and use relations from R for the transformations $w = w_0 \rightarrow \cdots \rightarrow w_t = w'$; $s(n)$ bounds from above the tape space (or computer memory) sufficient to implement all such transitions $w \rightarrow \cdots \rightarrow w'$. One of the results obtained is the following criterion: A finitely generated semigroup S has decidable word problem of polynomial space complexity if and only if S is a subsemigroup of a finitely presented semigroup H with polynomial space function.

Key words: generators and relations in semigroups, algorithm, space complexity, word problem

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1 Introduction

Let A be an alphabet, A^* the set of all words in A , and A^+ the set of non-empty words. We will use $|w|$ for the length of a word w , in particular the empty word 1 has length 0 . We write $S = \langle A \mid R \rangle$ for a semigroup (resp., monoid) presentation when $R \subset A^+ \times A^+$ (resp., $R \subset A^* \times A^*$).

Let S be a semigroup or monoid and $w, w' \in A^*$. A *derivation* of length $t \geq 0$ from w to w' , where $w, w' \in A^+$ or $w, w' \in A^*$, resp., is a sequence of words

$$w = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_t = w', \quad (1.1)$$

where “ $=$ ” denotes the letter-for-letter equality, and for $0 \leq i < t$, the word w_{i+1} results from w_i after a defining relation from R is applied, i.e., $w_i = ur'v, w_{i+1} = ur''v$ for some words u, v , and $(r', r'') \in R$ or $(r'', r') \in R$. Two words w, w' represent the same element of S (or they are equal in $S : w =_S w'$) iff there exists a derivation $w \rightarrow \cdots \rightarrow w'$.

The minimal (non-decreasing) function $d(n) : \mathbb{N} \rightarrow \mathbb{N}$ such that for every two words w, w' equal in S and having length $\leq n$, there exists a derivation (1.1) with $t \leq d(n)$, is called the Dehn function of the presentation $S = \langle A \mid R \rangle$ ([13], [2]). For *finitely presented* S (i.e., both sets A and R are finite), Dehn functions are usually taken up to equivalence to get rid of the dependence on a finite presentation for S (see [18]). To introduce this equivalence \sim , we write $f \preceq g$ if there is a positive integer c such that

$$f(n) \leq cg(cn) + cn \quad \text{for any } n \in \mathbb{N} \quad (1.2)$$

For example, we say that a function f is *polynomial* if $f \preceq g$ for a polynomial g . From now on, we use the following equivalence for nondecreasing functions f and g on \mathbb{N} .

$$f \sim g \quad \text{if both } f \preceq g \text{ and } g \preceq f \quad (1.3)$$

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It is not difficult to see that the Dehn function $d(n)$ of a finitely presented semigroup or monoid, or group S is recursive (or bounded from above by a recursive function) iff the word problem is algorithmically decidable for S (see [11], [6]). In this case, the word problem can be solved by a primitive algorithm that, given a pair of words w, w' of length $\leq n$, just checks if there exists a derivation (1.1) of length $\leq d(n)$. Therefore the nondeterministic *time* complexity of the word problem in S is bounded from above by $d(n)$. Moreover if Q is a finitely generated subsemigroup (submonoid, subgroup) of S , then one can use the rewriting procedure (1.1) for Q , and so the nondeterministic time complexity of the word problem for Q is also bounded by a function equivalent to $d(n)$.

A converse statement is also true. Assume that the word problem can be solved in a *finitely generated* semigroup S by a nondeterministic Turing machine (*NTM*) with time complexity $\leq T(n)$, where $T(n)$ is a superadditive function (i.e. $T(m+n) \geq T(m) + T(n)$). Then S is a subsemigroup of a *finitely presented* semigroup H with Dehn function $O(T(n)^2)$. This is proved in [2] while a similar statement for groups (but with the function $n^2 T(n^2)^4$ instead of $T(n)^2$) is obtained in [4]. As the main corollary, one concludes that the word problem in a finitely generated semigroup (group) H has time complexity of class *NP* (i.e., there *exists* a nondeterministic algorithm of polynomial time complexity, which solves the word problem for H) iff H is a subsemigroup (resp., subgroup) of a finitely presented semigroup (resp., group) with polynomial Dehn function.

Hence the notion of Dehn function is the (semi)group-theoretical counterpart of the concept of time complexity for algorithms. It turns out that the filling length functions introduced earlier in [13], [12], [3] (or briefly, space functions) of finitely presented groups are counterparts of the concept of space complexity of algorithms. The main theorem of [22] says that for a finitely generated group G such that the word problem in G is decidable by a deterministic Turing machine (*DTM*) with space complexity $f(n)$, there is an embedding of G in a finitely presented group H with space function equivalent to $f(n)$. In particular the following criterion is obtained: A finitely generated group H has decidable word problem of polynomial space complexity if and only if H is a subgroup of a finitely presented group G with a polynomial space function.

Thus, on the one hand, theorems from [2] and [4] provide a logical connection between Dehn functions of semigroups and groups and the time complexity of their word problems; and on the other hand, similar interrelation of space functions of groups and the space complexity is obtained in [22]. So it is natural to fill a gap regarding space functions of semigroups and the space complexity of the algorithmic word problem in semigroups.

In the present paper, we say that the derivation (1.1) has space $\max_{i=1}^t |w_i|$. For two words w and w' equal in $S = \langle A | R \rangle$, we denote by $space_S(w, w')$ the minimum of spaces of the derivations connecting w and w' , and define the value of the *space function* $s(n)$ to be equal to $\max(space(w, w'))$ over all pairs (w, w') of equal in S words with $|w|, |w'| \leq n$. An accurate definition of the space complexity (function) $f(n)$ for a Turing machine (*TM*) will be recalled in Subsection 2.1. Now we just note that the space complexities of machines are taken here up to the same equivalence 1.3 as the space functions of semigroups. Up to this equivalence, the time and space complexities of the word problem do not depend on the choice of a finite generator set, see [2], Prop. 2.1.

Theorem 1.1. *Let S be a finitely generated semigroup (monoid) such that the word problem in S is decidable by a *DTM* with space complexity $f(n)$. Then S is a subsemigroup (resp., submonoid) of a finitely presented monoid P with space function equivalent to $f(n)$.*

Remark 1.2. It follows from [18], [8] that even if S is finitely presented, one cannot define $P = S$ in Theorem 1.1. Baumslag's [1] 1-relator group $G = \langle a, b \mid (aba^{-1})b(aba^{-1})^{-1} = b^2 \rangle$ is a particular counterexample because the space function of G is not bounded from above by any multi-exponential function (see [10] and [23]) while the space (and time) complexity of the word problem in G is polynomial [19].

Corollary 1.3. *The word problem in a finitely generated semigroup (monoid) S is polynomial space decidable if and only if S is a subsemigroup (resp., submonoid) of a finitely presented monoid H with polynomial space function.*

We apply our approach to the realization problem: Which functions $f(n): \mathbb{N} \rightarrow \mathbb{N}$ are, up to equivalence, the space functions of finitely presented semigroups? It is not difficult to find examples of

(semi)groups with linear and exponential space functions, but it is not easy even to specify a (semi)group with space function n^2 .

Corollary 1.4. *The space complexity $f(n)$ of arbitrary DTM M is equivalent to the space function of some finitely presented semigroup (or monoid) P .*

This corollary reveals an extensive class of space functions of semigroups, including functions equivalent to $[\exp \sqrt[3]{n}]$, $[n^k]$ ($k \in \mathbb{N}$), $[n^k \log^l n]$, $[n^k \log^l(\log \log n)^m]$, etc. Note that we do not assume in the formulation of Theorem 1.4 that the function $f(n)$ is superadditive (i.e., $f(m+n) \geq f(m) + f(n)$) or grows sufficiently fast. (Compare with theorems in [26] and [2] on Dehn functions of groups and semigroups.) It follows, in particular, that there exists a finitely presented semigroup whose space function is not equivalent to any superadditive function. Recall that it is unknown if the Dehn function of arbitrary finitely presented group is equivalent to a superadditive function; see [14].

Corollary 1.5. *There is a finitely presented semigroup (and monoid) P with polynomial space complete word problem and with polynomial space function.*

We also describe the functions n^α which are (up to equivalence) space functions of semigroups (and monoids). As in [22], our approach is based on a modification of the proof of Savitch's theorem from [9] and the proof from [26], where the similar problem was considered for Dehn functions if $\alpha \geq 4$, and close necessary and sufficient conditions were obtained. (See also a dense series of examples with $\alpha \geq 2$ presented in [5].) Now we have $\alpha \geq 1$ in Corollary 1.6 below. Also it is worth to note that for space functions, the necessary and sufficient conditions just coincide.

To formulate the criterion, we call a real number α *computable with space $\leq f(m)$* if there exists a DTM which, given a natural number m , computes a binary rational approximation of α with an error $O(2^{-m})$, and the space of this computation $\leq f(m)$.

Corollary 1.6. *For a real number $\alpha \geq 1$, the function $[n^\alpha]$ is equivalent to the space function of a finitely presented semigroup (or monoid) iff α is computable with space $\leq 2^{2^m}$.*

It follows that functions $[n^\alpha]$ with any algebraic $\alpha \geq 1$ are all space functions of finitely presented semigroups (and monoids), as well as $[n^e]$, $[n^{\sqrt{\pi}}]$, etc.

Remark 1.7. Some of the above statements sound similar to propositions from [22], but they cannot be deduced from [22] since there, the set of admissible transformations in derivations is larger than here. (After Bridson and Riley [7], we also allowed cyclic permutations and fragmentations of words in [22].) It is an open question if Theorem 1.1 and its corollaries valid for groups as well, provided the derivations are based only on the applications of defining relations, as this is accepted in the present paper.

Recall that Higman proved in [15] that every recursively presented group is embeddable in a finitely presented one. A semigroup analog of this theorem was proved by Murskii in [20]. The approaches to groups and to semigroups are different, since in the group case one can use conjugations and HNN extensions to synchronize applications of all defining relations corresponding to one machine command (see [25]). For semigroup embeddings, Murskii [20] and Birget [2] use one tape symmetric input-output machines. We follow this line but one should look after the space complexity of the constructed machines. Moreover the *generalized* space complexity (see Subsection 2.1 for the definitions) of the latest modification M_5 must be equal to the space complexity of the initial machine M_0 . The trick used in Subsection 2.3 for this purpose, works if the machine M_0 is deterministic. (Therefore Theorem 1.1 is formulated in terms of deterministic space complexity while [4] and [2] consider only nondeterministic time complexity.)

As in [20] and [2], our embedding of S in a finitely presented semigroup P in Theorem 1.1 is based on the commands of the constructed machine. Unlike [20] and [2], now we should control the space function of P . Some additional technical difficulties appear because we want to obtain a monoid embedding (i.e., $1 \rightarrow 1$) if the semigroup S is a monoid. Monoid relation $v = 1$ is less convenient since the word v can be inserted before/after any letter of any word w . (Note that only semigroup embedding are under consideration in [2]. Whether the embedding from [20] is a monoid embedding or not, if S is a monoid, is also left inexplicit.)

In the remaining part of the proof we introduce derivation trapezia. They visualize derivations and make possible to use geometric images, e.g., bands, lenses, cups and caps. Furthermore, one can remove

unnecessary parts of trapezia (e.g., see Lemmas 3.7–3.11, 3.16). Such parts may not correspond to subderivation, and they can hardly be defined in a different language.

2 Machines

2.1 Definitions

We will use a model of *recognizing TM* which is close to the model from [26].

Recall that a *(multi-tape) TM with k tapes and k heads* is a tuple

$$M = \langle A, Y, Q, \Theta, \vec{s}_1, \vec{s}_0 \rangle$$

where A is the input alphabet, $Y = \sqcup_{i=1}^k Y_i$ is the tape alphabet, $Y_1 \supset A$, $Q = \sqcup_{i=1}^k Q_i$ is the set of states of the heads of the machine, Θ is a set of transitions (commands), \vec{s}_1 is the k -vector of start states, \vec{s}_0 is the k -vector of accept states. (\sqcup denotes the disjoint union.) The sets Y, Q, Θ are finite.

We assume that the machine normally starts working with states of the heads forming the vector \vec{s}_1 , with the head placed at the right end of each tape, and accepts if it reaches the state vector \vec{s}_0 . In general, the machine can be turned on in any configuration and turned off at any time.

A *configuration* of tape number i of a *TM* is a word uqv where $q \in Q_i$ is the current state of the head, u is the word to the left of the head, and v is the word to the right of the head, $u, v \in Y_i^*$. A tape is *empty* if u, v are empty words.

A *configuration* U of the machine M is a word

$$\alpha_1 U_1 \omega_1 \alpha_2 U_2 \omega_2 \dots \alpha_k U_k \omega_k$$

where U_i is the configuration of tape i , and the endmarkers α_i, ω_i of the i -th tape are special separating symbols.

An *input configuration* $w(u)$ is a configuration, where all tapes, except for the first one, are empty, the configuration of the first tape (let us call it the *input tape*) is of the form uq , $q \in Q_1$, u is a word in the alphabet A , and the states form the start vector \vec{s}_1 . The *accept configuration* is the configuration where the state vector is \vec{s}_0 , the accept vector of the machine, and all tapes are empty. (The requirement that the tapes must be empty is often removed for auxiliary machines which are used in construction of bigger machines.)

To every $\theta \in \Theta$, there corresponds a command (marked by the same letter θ), i.e., a pair of sequences of words $[V_1, \dots, V_k]$ and $[V'_1, \dots, V'_k]$ such that for each $j \leq k$, either both $V_j = uqv$ and $V'_j = u'q'v'$ are configurations of the tape number j , or $V_j = \alpha_j qv$ and $V'_j = \alpha_j q'v'$, or $V_j = uq\omega_j$ and $V'_j = u'q'\omega_j$, or $V_j = \alpha_j q\omega_j$ and $V'_j = \alpha_j q'\omega_j$ ($q, q' \in Q_j$).

In order to execute this command, the machine checks if V_i is a subword of the current configuration of the machine, and if this condition holds the machine replaces V_i by V'_i for all $i = 1, \dots, k$. Therefore we also use the notation: $\theta : [V_1 \rightarrow V'_1, \dots, V_k \rightarrow V'_k]$, where $V_j \rightarrow V'_j$ is called the j -th *part* of the command θ .

Suppose we have a sequence of configurations w_0, \dots, w_t and a word $h = \theta_1 \dots \theta_t$ in the alphabet Θ , such that for every $i = 1, \dots, t$ the machine passes from w_{i-1} to w_i by applying the command θ_i . Then the sequence $(w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_t)$ is said to be a *computation with history h* . In this case we shall write $w_0 \cdot h = w_t$. The number t will be called the *time* or *length* of the computation.

A configuration w is called *accepted* by a machine M if there exists at least one computation which starts with w and ends with the accept configuration. We do not only consider deterministic *TMs*, for example, we allow several transitions with the same left side.

A word u in the input alphabet A is said to be *accepted* by the machine if the corresponding input configuration is accepted. (A configuration with the vector of states \vec{s}_1 is never accepted if it is not an input configuration.) The set of all accepted words over the alphabet A is called the *language \mathcal{L}_M recognized by the machine M* .

If a DTM M halts on an input word $w \in A^*$ at a non-accepting state \vec{s} with all tapes empty, then one says that M *rejects* w . Speaking on deterministic *TM*, we will assume that every input configuration is

either accepted or rejected, i.e., M may not operate forever being switched on at an input configuration. In other words, we consider DTM-s M with recursive languages \mathcal{L}_M .

Let $|w_i|_a$ ($i = 0, \dots, t$) be the number of tape letters (or tape squares) in the configuration w_i . (As in [26], the tape letters are called *a-letters*.) Then the maximum of all $|w_i|_a$ will be called the *space of computation* $C : w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_t$ and will be denoted by $space_M(C)$. If $u \in A^*$ then, by definition, $space_M(u)$ is the minimal space of the computation that accepts or rejects the corresponding input configuration $w = w(u)$.

The number $S(n) = S_M(n)$ is the maximum of the numbers $space(u)$ over all words $u \in A^*$ with $|u| \leq n$. The function $S(n)$ will be called the *space complexity* of the DTM M .

The definition of the *generalized space complexity* $S'(n) = S'_M(n)$ is similar to the definition of space complexity but we consider arbitrary pair w_0, w_t of configurations which can be connected by a computation $w_0 \rightarrow \dots \rightarrow w_t$ (not just input configurations as in the definition of $S(n)$). We define $space_M(w_0, w_t)$ to be the minimal space of computations connecting w_0 and w_t , and $S'(n)$ is the minimal function that bounds from above all numbers $space_M(w_0, w_t)$ under the condition $|w_0|_a, |w_t| \leq n$. It is clear that $S(n) \leq S'(n)$.

2.2 Input-output machine

Assume that S is a semigroup (or monoid) generated by a finite set A , and the word problem in S is decidable by a DTM M_0 with space function $S_0(n)$. We define this more exactly as follows. The set of input words of M_0 consists of the words uv' , where u is a word in A and v' is a word in a disjoint alphabet A' which is a copy of A . Let u and v be two words over A , and $|u| + |v| \leq n$. Then (1) for a copy v' of v in A' , the word uv' belongs to the language \mathcal{L}_0 of M_0 iff $u =_S v$, (2) every input word of M_0 of length $\leq n$ is accepted or rejected with space $\leq S_0(n)$, and (3) $S_0(n)$ is the minimal function with Property (2).

However to obtain a Higman embedding of S into a finitely presented monoid we are not able to simulate the work of the recognizing machine M_0 by semigroup relations but following Murskii [20] and Birget [2] (and preserving the space complexity), we first transform it into an input-output machine M_1 . We will see that if u is an input word and v is an output word for a computation of M_1 , then $u =_S v$.

The definition of an input-output TM is similar to the definition of a recognizing TM, but the first tape is an “input-output tape” that holds the initial input and the final output. To define the space complexity $S(n)$ of an input-output machine, one consider input-output computations, where both the input word u and the output word v are of length at most n . The definition of the generalized space complexity $S'(n)$ is similar to the definition of space complexity but we consider arbitrary computations $w_0 \rightarrow \dots \rightarrow w_t$ with $\max\{|w_0|_a, |w_t|_a\} \leq n$, not just input-output computations as in the definition of $S(n)$.

We will assume that the words in A^* are ShortLex ordered.

Lemma 2.1. *There exists an input-output DTM M_1 such that*

- (a) *for every input word $u \in A^*$, there is an input-output computation C of M_1 with input u , and the output is the least word v equal to u in S ;*
- (b) *the space complexity $S_1(n)$ of M_1 is equivalent to the space complexity $S_0(n)$ of M_0 ;*
- (c) *depending on the state, any configuration w of the computation C contains either (i) a copy of the word u on one of the tapes or (ii) the output v on the input-output tape, and to obtain the output configuration $w(v)$ in Case (ii), it remains to erase all other tapes and accept; also we have $space(C) = S_1(|u|)$ in Case (ii);*
- (d) *a configuration with the start vector of states \vec{s}_1 cannot be reached after an application of a command of M_1 to any configuration.*
- (e) *if a configuration with vector of states \vec{s}_0 results after an application of a command of M_1 , then this command is the unique accepting command.*

Proof. The machine M_1 has two tapes more than M_0 . At first it writes a copy of u on an extra-tape T . Then it writes a current word $v \leq u$ (starting with the least v ; $v = 1$ if S is a monoid) on another extra-tape T' , and writes down the word uv' , where v' is a copy of v in a disjoint alphabet, on the input tape of M_0 (which is also the input-output tape of M_1). Then M_0 starts working to check whether $u =_S v$ or not. If “yes”, then M_1 rewrites v onto the output tape, cleans up all other tapes, and accepts.

Otherwise M_1 cleans up the tapes of M_0 , replaces the word v by the next word $v_+ \leq u$ on T' , and repeats the cycle with v_+ .

Since $u =_S u$, sooner or later the machine M_1 accepts u with Property (a). The first part of (c) follows from the above algorithm as well. Since the current word v is not longer than u and $|uv'| \leq 2|u|$, we have $S_{M_1}(n) \leq 3S_{M_0}(2n)$, and so $S_1(n) \preceq S_0(n)$. Now Property (b) and the second part of (c) follow from the inequality $S_0(n) \preceq S_1(n)$ which can be easily provided if one forces the machine M_1 to check *every* pair (\bar{u}, \bar{v}) with $|\bar{u}|, |\bar{v}| \leq |u|$ (even the shortest v_0 with $v_0 =_S u$ is already found). To obtain Property (d), it suffices to add special states for input configuration: the first command changes these states, and the state letters from \vec{s}_1 do not occur in other commands. Similarly, one obtains Property (e). \square

2.3 Machine with equal space complexity and generalized space complexity

In this subsection, we construct an NTM M_2 which inherits the basic characteristics of the DTM M_1 and has equivalent generalized space complexity and space complexity. For this goal we adapt the approach from [22] to input-output machines.

Assume that M_1 has k tapes, and let its first tape be the input-output tape. Then we add a tape numbered $k+1$, which is empty for input/output configurations, and we organize the work of the 3-stage machine M_2 as a sequential work of the following machines M_{21} , M_{22} , and M_{23} .

The machine M_{21} uses only one command θ_* that does not change states and adds one square with an auxiliary letter $*$ to the $(k+1)$ -st tape, i.e., the command θ_* has the form

$$[q_1\omega_1 \rightarrow q_1\omega_1, \alpha_2q_2\omega_2 \rightarrow \alpha_2q_2\omega_2, \dots, \alpha_kq_k\omega_k \rightarrow \alpha_kq_k\omega_k, q_{k+1}\omega_{k+1} \rightarrow *q_{k+1}\omega_{k+1}]$$

The machine M_{21} can execute this command arbitrarily many times while the tapes numbered $1, \dots, k$ keep the copy of an input configuration of M_1 unchanged. Then a connecting rule $\theta_{12} : [q_1 \rightarrow q'_1\omega_1, \dots, \alpha_kq_k\omega_k \rightarrow \alpha_kq'_k\omega_k, q_{k+1}\omega_{k+1} \rightarrow q'_{k+1}\omega_{k+1}]$ changes all states of the heads and switches on the machine M_{22} . Here $(q'_1, \dots, q'_k) = \vec{s}_1$ is the vector of start states for M_1 .

The work of M_{22} on the tapes with numbers $1, \dots, k$ copies the work of M_1 . But the extension θ' of every command θ of M_1 to the $(k+1)$ -st tape is defined so that its application does not change the current space. More precisely, if a command θ inserts m_1 tape squares and deletes m_2 tape squares on the first k tapes, then θ' inserts $m_2 - m_1$ (deletes $m_1 - m_2$) squares with letter $*$ on the $(k+1)$ -st tape if $m_1 - m_2 \leq 0$ (if $m_1 - m_2 \geq 0$). That is the $(k+1)$ -st component of θ' has the form $q_{k+1}\omega_{k+1} \rightarrow *^{m_2-m_1}q_{k+1}\omega_{k+1}$ (resp., $*^{m_1-m_2}q_{k+1}\omega_{k+1} \rightarrow q_{k+1}\omega_{k+1}$). Note that one cannot apply θ' if $m_1 - m_2$ exceeds the current number of squares on the tape numbered $k+1$.

The connecting command θ_{23} is applicable when M_{22} reaches the output configuration on the first k tapes. It changes the states and switches on the machine M_{23} erasing all squares on the $(k+1)$ -st tape (one by one).

Let w be a configuration of the machine M_2 such that $w \cdot \theta_*$ is defined, or such that w is obtained after an application of the connecting command θ_{12} . Then we have an input configuration on the tapes with numbers $1, \dots, k$ (plus several $*$ -s on the $(k+1)$ -st tape). We will denote by $u(w)$ the input word u written on the first tape. It is an input word for the machine M_1 as well, and the expression $space_{M_1}u(w)$ makes sense.

The connecting commands θ_{12} and θ_{23} are not invertible in M_2 by definition. Therefore every non-empty computation of M_2 has history of the form $h_1h_2h_3$ or h_1h_2 , or h_1 , or h_2h_3 , or h_3 , where h_l is the history for M_{2l} , ($l = 1, 2, 3$). (To simplify notation we attribute the command θ_{12} (the command θ_{23}) to h_2 (to h_3).)

Lemma 2.2. (a) For every input word u , the machine M_1 and M_2 give out the same output v . (b) The space complexity $S_2(n)$ and the generalized space complexity $S'_2(n)$ of M_2 are both equivalent to $S_1(n)$.

Proof. Assume that u is converted to the output word v by M_1 . Then u can be converted to v by M_2 as well because the machine M_{21} can insert sufficiently many squares (equal to $space_{M_1}(u) - |u|$) so that the input-output computation of M_1 can be simulated by M_{22} . Also it is clear from the definition of M_2 , that every accepting computation for M_2 having a history $h_1h_2h_3$ as above, simulates, at stage 2, an accepting computation of M_1 with history h_2 . This proves Statement (a) and equality $S_1(n) = S_2(n)$.

Assume now that $C : w = w_0 \rightarrow \dots \rightarrow w_t = w'$ is a computation of M_2 with minimal space for given w and w' , and $h \equiv h_1 h_2 h_3$ is the history with the above factorization (some of the factors h_i can be empty here). If the word h_1 is empty, then $|w_0| \geq \dots \geq |w_n|$ by the definition of the machines M_{22} and M_{32} . Hence the space of this computation is equal to $|w|_a$. Similarly, it is $|w'|_a$ if h_3 is empty. Then let both h_1 and h_3 be non-empty. It follows that the machine M_2 starts (ends) working with a copy of an input (resp., output) configuration of the machine M_1 , i.e., the input-output tape of this configuration contains an input word $u = u(w)$ (output word $v = v(w')$) and the additional $(k+1)$ -st tape has m squares (resp., m' squares) for some $m \geq 0$. We consider two cases.

Case 1. Suppose $m \geq \text{space}_{M_1}(u) - |u|$. This inequality says that the additional tape has enough squares to enable M_{22} to simulate the computation of M_1 with the input word u . Hence there is an M_2 -computation $w_0 \rightarrow \dots \rightarrow w_{n'}$ with history of the form $h'_2 h'_3$, and so its space, as well as the space of our original computation, is $|w|_a$.

Case 2. Suppose $m < \text{space}_{M_1}(u) - |u|$. Then there is a computation $w_0 \rightarrow \dots \rightarrow w_{n'}$ such that the commands of its M_{21} -stage insert squares until the total number of squares of the $(k+1)$ -st tape becomes equal to $\text{space}_{M_1}(u) - |u|$, and then the machines M_{22} and M_{23} work in their standard manner. The space of this (and the original) computation is $\text{space}_{M_1}(u)$.

The estimates obtained in cases 1 and 2 show that $S'_2(n) \leq \max(S_1(n), n)$. Hence

$$S_1(n) = S_2(n) \leq S'_2(n) \leq \max(S_1(n), n) \sim S_1(n),$$

and statement (b) is completely proved too. \square

2.4 Symmetric machine M_3

For every command θ of a TM , given by a vector $[V_1 \rightarrow V'_1, \dots, V_k \rightarrow V'_k]$, the vector $[V'_1 \rightarrow V_1, \dots, V'_k \rightarrow V_k]$ also gives a command of some TM . These two commands θ and θ^{-1} are called *mutually inverse*.

Since the machine M_1 is deterministic, the machine M_2 has no invertible commands at all. The definition of the *symmetric* machine $M_3 = M_2^{\text{sym}}$ is the following. Suppose $M_2 = \langle X, Y, Q, \Theta, \vec{s}_1, \vec{s}_0 \rangle$. Then by definition, $M_2^{\text{sym}} = \langle X, Y, Q, \Theta^{\text{sym}}, \vec{s}_1, \vec{s}_0 \rangle$, where Θ^{sym} is the minimal *symmetric* set containing Θ , that is, with every command $[V_1 \rightarrow V'_1, \dots, V_{k+1} \rightarrow V'_{k+1}]$ it contains the inverse command $[V'_1 \rightarrow V_1, \dots, V'_{k+1} \rightarrow V_{k+1}]$; in other words, $\Theta^{\text{sym}} = \Theta^+ \sqcup \Theta^-$, where $\Theta^+ = \Theta$ (the set of positive commands) and $\Theta^- = \{\theta^{-1} \mid \theta \in \Theta\}$ (the set of negative commands).

A computation $w_0 \rightarrow \dots \rightarrow w_t$ of M_3 (or other machine) is called *reduced* if its history is a reduced word. If the history $h = \theta_1 \dots \theta_t$ contains a subword $\theta_i \theta_{i+1}$, where the commands θ_i and θ_{i+1} are mutually inverse, then obviously there is a shorter computation $w_0 \rightarrow \dots \rightarrow w_{i-1} = w_{i+1} \rightarrow \dots \rightarrow w_t$ whose space does not exceed the space of the original one.

Lemma 2.3. *Let $C : w_0 \rightarrow \dots \rightarrow w_t$ be a reduced computation of M_3 with history $h = \tau h' \tau'$, where $\tau, \tau' \in \{\theta_{12}^{\pm 1}, \theta_{23}^{\pm 1}\}$ and every command from h' is a command of M_{22} or its inverse. Then the words $u = u(w_0)$ and $v = u(w_t)$ are equal in S (recall that $u(w)$ is the subword of w written on the input-output tape), and $\text{space}(C) \geq S_1(\max(|u|, |v|))$. Furthermore, if $\tau' = \theta_{23}$, then $\tau = \theta_{12}$ and the word h' is positive.*

Proof. Note that h' has no 2-letter subwords $\theta^{-1} \theta'$, where both θ and θ' are positive commands of M_{22} since then different commands θ and θ' would be applicable to the same configuration $w_i \cdot \theta^{-1} = w_0 \cdot (\dots \theta^{-1})$, and so the corresponding commands of M_1 would be also applicable to the same configuration contrary to the determinism of M_1 . Hence $h' = g_1 g_2^{-1}$, where both g_1 and g_2 are (positive) histories for M_{22} .

Since one may replace C by the inverse computation, it suffices to consider three cases: (a) $\tau = \theta_{12}, \tau' = \theta_{23}$, (b) $\tau = \theta_{12}, \tau' = \theta_{12}^{-1}$, and (c) $\tau = \theta_{23}^{-1}, \tau' = \theta_{23}$.

Case (a). In this case g_2 is empty since θ_{23} can be applied only after the unique command of M_{22} corresponding to the accepting command of M_1 (see Lemma 2.1 (e)). Thus h' is the history of an M_{22} -computation, and by Lemma 2.2(a), the corresponding M_1 -computation converts u into v . So $u =_S v$ by Lemma 2.1 (a). We also have $\text{space}(C) \geq S_1(\max(|u|, |v|))$ by Lemma 2.1(c), since $v \leq u$ in this case by the definition of M_1 .

Case (c). The same argument shows now that both g_1 and g_2 are empty, a contradiction. Case (c) is impossible.

Case (b). Note that both g_1 and g_2 are non-empty since a (positive) M_{22} -command cannot follow by θ_{12}^{-1} by the Property (d) from Lemma 2.1 since the machine M_{22} copies M_1 . If $u = v$, then two M_1 -computations C_1 and C_2 corresponding to the M_{22} -computations $w_0 \rightarrow \dots \rightarrow w_0 \cdot g_1$ and $w_t \rightarrow \dots \rightarrow w_t \cdot g_2$ have equal the first and the last configurations. Since M_1 is deterministic it follows that $C_1 = C_2$. But g_1 and g_2 are completely determined by their M_1 -parts C_1 and C_2 . Hence we have $g_1 = g_2$, a contradiction. Thus $u \neq v$.

Now by the inequality $u \neq v$ and Lemma 2.1 (c), the configuration $w_0 \cdot g_1 = w_t \cdot g_2$ must contain (the same) output word on the same input-output tape for the input words u and v of M_1 . We also have $u =_S v$ by Lemmas 2.1(a) and 2.2. Furthermore, by Lemma 2.1 (c), the spaces of C_1 and C_2 are at least $S_1(|u|)$ and $S_1(|v|)$, respectively. Therefore $\text{space}(C) \geq S_1(\max(|u|, |v|))$.

The claims of the lemma are proved. □

We say that a computation $w_0 \rightarrow \dots \rightarrow w_t$ is an input-input computation of the machine M_3 if both w_0 and w_t are input configurations of M_2 (and of M_3 as well).

Lemma 2.4. (a) For two input configurations w and w' , of M_3 , there exists an input-input computation $w \rightarrow \dots \rightarrow w'$ iff $u(w) =_S u(w')$. If $w \neq w'$, then $\text{space}_{M_3}(w, w') = S_1(\max(|u(w)|, |u(w')|))$.

(b) Let $C : w \rightarrow \dots \rightarrow w'$ be a reduced input-output computation of M_3 with $u(w) = u$. Then $\text{space}(C) \geq S_1(|u|)$.

Proof. (a) Assume that $u(w) =_S u(w')$, and let v be the least word equal to $u(w)$ (and to $u(w')$) in S . By the definition of M_1 , we have a computation C_1 of M_1 connecting the input configuration of M_1 with input words $u = u(w')$ and the output configuration of M_1 with output v . By Lemma 2.1 (c), $\text{space}(C_1) = S_1(|u|)$. Similarly we have C_2 with input word $u' = u(w')$ and the same output word v . Let C_3 and C_4 , resp., be the corresponding computations of M_2 (see Subsection 2.3). Define C' to be the reduced form of the computation $C_3 C_4^{-1}$ of M_3 . Then C' connects w and w' and $\text{space}(C') \leq S_1(\max(|u|, |u'|))$.

Assume now that $u = u(w) \neq u' = u(w')$ and C is an input-input computation $w \rightarrow \dots \rightarrow w'$ of M_3 . The history of C is $h \equiv h_0 \tau_1 h_1 \tau_2 \dots \tau_s h_s$, where $\tau_i \equiv \theta_{12}^{\pm 1}$ or $\tau_i \equiv \theta_{23}^{\pm 1}$ for $i \leq s$, and the subwords h_i -s contain no connecting commands. The connecting commands τ_i -s and the subcomputations with histories h_i -s whose commands correspond to the commands of M_{21} or to the commands of M_{23} (or to inverses) do not change the content of the input-output tape. By Lemma 2.3, the subcomputations of the form $\tau_{i-1} h_i \tau_i$, where h_i corresponds to M_{22} , do not change the content of the input-output tape modulo the relations of S . Since h_0 and h_s must consist of the commands of M_{21} (or inverses) for an input-input computation, we obtain $u =_S u'$, as required.

Furthermore, if $u \neq u'$, then $s > 1$, and the subcomputations with histories $\tau_1 h_1 \tau_2$ and $\tau_{s-1} h_{s-1} \tau_s$ satisfy the assumption of Lemma 2.3, whence $\text{space}(C) \geq \max S_1(|u|, |u'|)$.

The obtained inequalities for $\text{space}(C)$ and $\text{space}(C')$ complete the proof of Statement (a).

(b) Consider the history $h \equiv h_0 \tau_1 h_1 \tau_2 \dots \tau_s h_s$ of C . Since C is a reduced input-output computation, we have that h_0 must consist of (positive) commands of M_{21} , $\tau_1 = \theta_{12}$, and $\tau_s = \theta_{23}$. So $u = u(w) = u(w \cdot h_0)$, $s \geq 2$, and Statement (b) follows from Lemma 2.3 applied to the subcomputation with history $\tau_1 h_1 \tau_2$. □

The notation $\text{space}(C) \preceq f(n)$, where $n = n(C)$ depends on the computation C , will mean further that for some constants c_1, c_2, c_3 independent of C , we have $\text{space}(C) \leq c_1 f(c_2 n) + c_3 n$.

Lemma 2.5. Let $C : w_0 \rightarrow \dots \rightarrow w_t$ be a computation of M_3 with the smallest space for the fixed w_0 and w_t . Then $\text{space}(C) \preceq S_1(\max(|w_0|_a, |w_t|_a))$.

Proof. Let us say that a configuration w of M_3 has type 1 (resp., 2 or 3) if a command of M_{21} (resp., of M_{22} or M_{23}) or its inverse is applicable to w .

Case 1. Assume that both w_0 and w_t are of type 1. Then using the command inverse to the command of M_{21} , we can start with w_0 and clean up the tape number $k+1$ preserving the content $u(w_0)$ of the input-output tape: $C_1 : w_0 \rightarrow \dots \rightarrow \bar{w}$. Similarly we have $C_2 : w_t \rightarrow \dots \rightarrow \bar{\bar{w}}$. The spaces

of these computations are $|w_0|_a$ and $|w_t|_a$, resp. The input configurations \bar{w} and $\bar{\bar{w}}$ can be connected by a computation $C_1^{-1}CC_2$, and by Lemma 2.4, there exists an input-input computation $C_3 : \bar{w} \rightarrow \dots \rightarrow \bar{\bar{w}}$ of space $S_1(\max(|u(\bar{w})|, |u(\bar{\bar{w}})|)) = S_1(\max(|u(w_0)|, |u(w_t)|))$. The same upper bound holds for the computation $C_1C_3C_2^{-1} : w_0 \rightarrow \dots \rightarrow w_t$, which proves the lemma in this case.

Case 2. Assume that w_0 and w_t have types 1 or 3. Taking into account the previous case, we may assume that w_t is of type 3. If all w_i -s have type 3 in C , then their lengths monotonically increase or decrease since M_{23} has only one command. Hence $\text{space}(C) \leq \max(|w_0|_a, |w_t|_a)$. Otherwise, by Lemma 2.3, the history of C must have a suffix $\theta_{21}h'\theta_{23}h''$, where h'' consists of the commands of M_{23} and h' is a history of an M_{22} -computation. It follows from the definition of M_{22} that $v = u(w_t)$ is an output word.

By the definition of M_2 , there exists an M_2 -computation C' which starts with an input configuration w' with the input word v , ends with the output configuration with output also v , and has space $S_2(|v|)$. Also there is a computation C'' of M_{23} which deletes several letter in w_t and ends with the same configuration as C' . Hence the computation $C''C'^{-1}$ converts the configuration w_t into an input configuration w' of length $\leq |w_t|$, and has space $\max(S_2(|v|), |w_t|_a) \preceq S_2(|w_t|_a) \sim S_1(|w_t|_a)$ by Lemma 2.2 (b). Hence it suffices to obtain a desired upper estimate for $\text{space}_{M_3}(w_0, w')$. But now w' is of type 1. Similarly, if w_0 is of type 3, it can be replaced by a word w'' of type 1. Thus Case 2 reduces to Case 1.

Case 3. One of the words w_0, w_t (or both) is of type 2. If all w_i -s in C are of type 2, then the commands from C do not change the lengths, and it is nothing to prove. Otherwise one can find $j > i$ such that w_0, \dots, w_i have equal lengths, w_j, \dots, w_t are of the same length too, and the subcomputation $w_i \rightarrow \dots \rightarrow w_j$ satisfies the assumptions of Case 1 or of Case 2. This completes the proof. \square

Lemma 2.6. *The space function $S_3(n)$ and the generalized space function $S'_3(n)$ of the machine M_3 are both equivalent to the space function $S_1(n)$ of M_1 .*

Proof. By Lemma 2.5, we have $S'_3(n) \preceq S_1(n)$. On the other hand, by the definition of $S_1(n)$, there is an input-output computation $C : w_0 \rightarrow \dots \rightarrow w_t$ of M_1 with the same input and output words $u = u(w_0) = u(w_t)$ of length n , and with space $S_1(|u|) = S_1(n)$. Then one can construct a computation $C' : w'_0 \rightarrow \dots \rightarrow w'_t$ of M_2 (and of M_3) of the same space $S_1(n)$, where the M_{22} -portion of C' corresponds to C . By Lemma 2.4 (b), any computation of M_3 connecting w'_0 and w'_t has space at least $S_1(|u(w'_0)|) = S_1(|u|) = S_1(n)$, whence $S'_3(n) \geq S_1(n)$, and the statement of Lemma 2.6 is proved. \square

A configuration w of a machine M is called *reachable* if there is a computation $w_0 \rightarrow \dots \rightarrow w_t = w$, where w_0 is an input configuration of M .

Lemma 2.7. *If w is a reachable configuration of the machine M_3 , then there is a computation $w_0 \rightarrow \dots \rightarrow w_t = w$, where w_0 is an input configuration and $|w_0| \leq |w|$.*

Proof. By definition, we have a computation $C : w_0 \rightarrow \dots \rightarrow w_t = w$ starting with an input configuration w_0 . We will induct on $|w_t|_a$, and for fixed $|w_t|_a$, we will induct on t . The base $|w_t|_a = t = 0$ is obvious, and moreover, we may always assume that $t > 0$.

If $w_{t-1} \rightarrow w_t$ is a transition of the machine M_{21} or its inverse, then one can obtain an input configuration w'_0 from $w_t = w$ using a repeated erasing of the auxiliary letter $*$ by the command θ_*^{-1} . Clearly we have $|w'_0| \leq |w|$, and the statement is true.

If $w_{t-1} \rightarrow w_t$ is a transition of the machine M_{22} or its inverse, then $|w_{t-1}|_a = |w_t|_a$, and it remains to apply the inductive hypothesis to the reachable word w_{t-1} .

Now we assume that $w_{t-1} \rightarrow w_t$ is a transition of the machine M_{23} or its inverse. By Lemma 2.3, the history of C must have a suffix of the form $\theta_{12}h'\theta_{23}h''$, where h' (resp., h'') is a product of the commands of M_{22} (of the commands of M_{23} or inverses). Since the subcomputation $C' : w_r \rightarrow \dots \rightarrow w_s$ with history h' is an input-output computation of M_{22} , there is a computation of M_1 with the input $u = u(w_r)$ and the output $v = u(w_s)$. By Lemma 2.1 (a), the word v is not equal in S to a shorter word. Therefore, by the definition of M_1 , there is an input-output computation of M_1 with both input and output words equal to v . Then by Lemma 2.2 (a), there is a computation of M_2 (and M_3) with the input and the output equal to v ; it starts with a configuration w' , where $u(w') = v$ and ends with some w'' , with $u(w'') = v$. Note that $v = u(w_s) = u(w_t)$. Therefore $|w'|_a = |v| \leq |w_t|_a$, and so $|w'| \leq |w_t|$. The command θ_{23} is applicable to both configurations w'' and w_s , and so they can be connected by an M_3 -computation,

where every command (or its inverse) is a command of M_{23} . It follows that there is an M_3 -computation $w' \rightarrow \dots \rightarrow w'' \rightarrow \dots \rightarrow w_s \rightarrow \dots \rightarrow w_t$, and the lemma is proved. \square

Below we will treat the NTM M_3 as a nondeterministic ‘input-input’ machine, i.e., the ‘purpose’ of M_3 is to transform an input configuration $w(u)$ to an input configuration $w(v)$. Consider the following relation $u \sim v$ on the set of words in the input alphabet: *there exists a (reduced) input-input M_3 -computation $w(u) \rightarrow \dots \rightarrow w(v)$* . This is an equivalence relation. Indeed, it becomes reflexive if one adds computations of length 0. The transitivity is obvious, and its symmetry follows from the symmetry of M_3 . So we use the term *equivalence machine* (or just *E-machine*) for a symmetric input-input NTM.

2.5 One-tape machine

It is well known that any NTM is equivalent to a one-tape NTM with the same space complexity (see Corollary 1.16 in [9]). But here we take some precautions to preserve the *generalized* space complexity.

Let M be a k -tape E-machine with an input alphabet A . We will construct an equivalent (i.e., defining the same equivalence relation on the words over A) one-tape E-machine M' . M' has the same input alphabet A , and at the preliminary stage it inserts the endmarkers $\alpha_1, \dots, \omega_k$ and the components q_{11}, \dots, q_{1k} of the vector of start states \vec{s}_1 of M , that is, these letters become tape letters of M' , and at the first stage, M' converts an input configuration $\alpha u q_1 \omega$ of M' into the configuration $\alpha \alpha_1 u q_{11} \omega_1 \dots \alpha_k q_{1k} \omega_k q_1 \omega$ (i.e., the head of M' runs to α , check the left endmarker, inserts the letter α_1 , and then returns to ω inserting the remaining extra-letters $q_{11}, \omega_1, \dots, \alpha_k, q_{1k}, \omega_k$).

Every configuration w of M is represented by the configuration $W = \alpha w q \omega$ of M' , where the state letter q of M' is the vector $(q^{(1)}, \dots, q^{(k)})$ of states of w (but all the letters of w , including the extra-letters, are tape letters for M'). For every transition $w \rightarrow w \cdot \theta$ of M with positive command θ , we construct a computation $C(w, \theta) : W \rightarrow \dots \rightarrow W'$ of M' as follows. The first command just changes the state q by the state q_θ , i.e., it memorizes θ , and M' will remember θ until the computation $C(w, \theta)$ ends. This command involves the endmarker ω . Then the head of M' goes to the left and simulates the application of the command θ when it meets $q^{(i)}$. For example, if the i -th part of θ is $cq^{(i)}a \rightarrow dq^{(i)}b$, then the corresponding computation of M' is of the form

$$\dots cq^{(i)}q_\theta(1)a \dots \rightarrow \dots cq^{(i)}q_\theta(2)b \dots \rightarrow \dots cq_\theta(3)q^{(i)}b \dots \rightarrow \dots dq_\theta(4)q^{(i)}b \dots,$$

where $q_\theta(1), \dots, q_\theta(4)$ are auxiliary state letters of M' . So the head of M' must reach α (there is a command involving α) and then returns to ω . The last command of this computation $q'_\theta \omega \rightarrow q' \omega$ forgets θ , and the state letter q' of W' is just the the vector of states of the configuration w' , so that W' corresponds to w' .

Remark 2.8. Two different positive (or two different negative) commands of M' cannot be applicable to a configuration containing a state letter indexed by some θ .

Lemma 2.9. (a) *The E-machines M and M' recognize the same equivalence relation on the set of input words.*

(b) *They have equivalent generalized space functions $S'_M(n)$ and $S'_{M'}(n)$.*

(c) *If $M = M_3$ and W is a reachable configuration of M' , then there is a computation $W_0 \rightarrow \dots \rightarrow W$, where W_0 is an input configuration of M' and $|W_0|_a \leq |W|_a + c$ for a constant c independent of W .*

(d) *If a reduced computation $W_0 \rightarrow \dots \rightarrow W_t$ of M' has no commands involving α or has no commands involving ω , then t is bounded from above by $c_1|W_0| + c_2$ for some constants c_1, c_2 .*

(e) *If a computation of the form $Uq_1\omega = W_0 \rightarrow \dots \rightarrow W_t$, where q_1 is the start state of M' , has no commands involving α then $|W_0| = |W_t|$ and the computation commands involve tape letters only from the input alphabet. A non-empty computation of the form $Uq_1\omega \rightarrow \dots \rightarrow U'q_1\omega$ has a command involving α .*

(f) *Let an M' -computation starts with an input configuration $w_0 = \alpha u q_1 \omega$ and ends with $w_t = \alpha u' q_1 \omega$. Then w_t is also an input configuration.*

Proof. Observe that the computation $C(w, \theta) : W \rightarrow \dots$ exists iff one may apply θ to w and W is the configuration of M' corresponding to w . Moreover, if two configurations W and W' of M' represent some configurations w and w' of M , then they can be connected by a computation $C(w, \theta)$ iff $w' = w \cdot \theta$.

(a) By the definition of M' , every input-input computation of M can be simulated by M' . Now let us consider a non-empty reduced input-input computation $C' : W_0 \rightarrow \dots \rightarrow W_t$ of M' , and denote by W_{i_1}, \dots, W_{i_s} ($0 < i_1 < \dots < i_s < t$) the intermediate configurations representing the configurations of M (i.e., M' do not remember the commands of M in these states). Since there are no other configurations with this property between $W_{i_{m-1}}$ and W_{i_m} , all the commands of the subcomputation $W_{i_{m-1}} \rightarrow \dots \rightarrow W_{i_m}$ must correspond to the same command θ of M . By Remark 2.8, the history of this subcomputation has no subwords of the form $\tau^{-1}\tau'$ (of the form $\tau'\tau^{-1}$), where both τ and τ' are positive commands of M' . Therefore the subcomputation must be of the form $C(w_{m-1}, \theta)$ or $C(w_m, \theta)^{-1}$ for a positive command θ of the machine M , and $W_{i_{m-1}}, W_{i_m}$ correspond to w_{m-1} and to $w_m = w_{m-1} \cdot \theta$, resp., or to $w_{m-1} = w_m \cdot \theta$ and to w_m , resp.

Since the preliminary stage $W_0 \rightarrow \dots \rightarrow W_{i_1}$ (resp., $W_t \rightarrow \dots \rightarrow W_{i_s}$) is also deterministic, the pair of input words for C' coincides with the pair of input words for the computation $C : w_1 \rightarrow \dots \rightarrow w_s$ of M , and so M and M' recognize the same binary relation.

(b) Let now $C' : W_0 \rightarrow \dots \rightarrow W_t$ be an arbitrary reduced computation of M' with $\max(|W_0|_a, |W_t|_a) \leq n$. We define W_{i_1}, \dots, W_{i_s} as in Part (a) of the proof. If $s = 0$, then for every j , $|W_j|_a \leq n + c$ for a constant c independent of the computation since the computations of the form $C(w, \theta)$ and the preliminary computations (and their subcomputations), up to a constant, do not change the space. So we will assume that $s \geq 1$. Then as in Part (a), the computation $C' : W_{i_1} \rightarrow \dots \rightarrow W_{i_s}$ corresponds to a computation $w_1 \rightarrow \dots \rightarrow w_s$ of M , where $|w_j|$ and $|W_{i_j}|$ are almost (up to an additive constant) equal. Therefore w_1 and w_s can be connected by a computation C of M of space at most $S'_M(n + c)$. There is a computation C''' of M' corresponding to C and having almost the same space. If we replace the subcomputation C'' of C' by C''' we get a computation of M' which connects W_0 and W_t and has space $\leq S'_M(n + c) + c$. Hence $S'_{M'}(n) \leq S'_M(n)$.

Similarly, if we start with a computation $C : w_1 \rightarrow \dots \rightarrow w_s$ of M with $|w_1|_a, |w_s|_a \leq n$, then we can replace it by a computation of M of space at most $S'_M(n + c)$, whence $S'_M(n) \leq S'_{M'}(n)$, as required.

(c) Assume now that $W_t = W$ and W_0 is an input configuration in the computation C' from (b). Now we consider the computation $C : w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_s$ of M , where w_j ($j \leq s$) corresponds to the configuration W_{i_j} of M' . By Lemma 2.7, one can find a computation $w'_0 \rightarrow \dots \rightarrow w'_{s'} = w_s$ of M , such that w'_0 is an input configuration and $|w'_0| \leq |w'_{s'}|$. Then one can construct a computation $W'_0 \rightarrow \dots \rightarrow W'_{i'_1} \rightarrow \dots \rightarrow W'_{i'_{s'}}$, where each $W'_{i'_j}$ represents w'_j , and therefore $|W'_0| \leq |W'_{i'_{s'}}|$. This give a computation

$$W'' \rightarrow \dots \rightarrow W'_0 \rightarrow \dots \rightarrow W'_{i'_{s'}} = W_{i_s} \rightarrow \dots \rightarrow W_t,$$

where W'' is an input configuration of M' , and $|W''| \leq |W_t| + c$, as required, because the preliminary subcomputation $W'' \rightarrow \dots \rightarrow W'_0$ does not decrease the space and the subcomputation $W_{i_s} \rightarrow \dots \rightarrow W_t$ is either empty or a part of a computation $C(w_s, \theta)$, and therefore it can remove a bounded number of tape letters.

(d) Follows from the fact that the computation of the form $C(w, \theta)$ involves both α and ω .

(e) The (reduced) work of M' is deterministic in the beginning: the head goes to the left until it reaches and checks the endmarker α . This implies Property (e).

(f) We must show that u' is a word in the input alphabet A . For this goal we can (1) assume that the computation is reduced, (2) consider the inverse computation $w_t \rightarrow \dots \rightarrow w_0$, and (3) take into account that at the preliminary stage, the machine M' verifies (when the head goes to α) if all the letters of the tape word belong to A .

□

We will use the doubling of the tape alphabet. This is a well-known trick helpful for simulating of machine commands by (semi)group relations (see [25]). Let Y be a tape alphabet of a one-tape machine M . We denote by Y_l and Y_r two disjoint copies of Y ('left' and 'right') and replace every configuration $\alpha u q v \omega$ of M by $\alpha u_l q v_r \omega$, where u_l (resp., v_r) is a copy of u in Y_l (in Y_r). Respectively, one modifies every command, e.g., a command $a q b \rightarrow c q' d$ is replaced by $a_l q b_r \rightarrow c_l q' d_r$. The input alphabet is replaced by its copy $A_l \subset Y_l$. Clearly, one obtain one-to one correspondence between the computations of M and the computations of the modified TM. The constructed machine inherits the basic properties of M . In particular, it has the same generalized space function.

Lemma 2.10. Assume that a multi-tape DTM M_0 solves the word problem in a finitely generated semi-group or monoid S with space function $S_0(n)$. Then there is a one-tape E-machine M_5 such that

- (a) the equivalence relation recognized by M_5 is the set of all pairs of words (u, v) in the generators of S satisfying the equality $u =_S v$;
- (b) the generalized space function $S'_5(n)$ of M_5 is equivalent to $S_0(n)$;
- (c) the left and right parts of the tape alphabet of M_5 are disjoint;
- (d) if w is a reachable configuration of M_5 , then there is a computation $w_0 \rightarrow \dots \rightarrow w$, where w is an input configuration of M_5 and $|w_0|_a \leq |w|_a + c$ for an integer $c \geq 1$ independent of w ;
- (e) if a reduced computation $w_0 \rightarrow \dots \rightarrow w_t$ of M_5 has no commands involving α or has no commands involving ω , then t is bounded from above by $c_1|w_0| + c_2$ for some constants c_1, c_2 ;
- (f) if a computation $W_0 = Uq_1\omega \rightarrow \dots \rightarrow W_t$ of M_5 has no commands involving α , then $|W_0| = |W_t|$ and the commands of this computation do not involve letters from $Y_l \setminus A_l$. A non-empty reduced computation $Uq_1\omega \rightarrow \dots \rightarrow U'q_1\omega$ of M_5 has a command involving α .
- (g) let an M_5 -computation starts with an input configuration $w_0 = \alpha u q_1 \omega$ and ends with $w_t = \alpha u' q_1 \omega$. Then w_t is also an input configuration of M_5 .

Proof. Recall that starting with the DTM M_0 we have constructed the input-output DTM-s M_1, M_2 , and an E-machine M_3 . Let us use the construction of this subsection assuming that $M_3 = M$ and $M_4 = M'$. Doubling the tape alphabet we get a machine M_5 providing Property (c). Then the statement (a) follows from Lemmas 2.4 (a), 2.9 (a), and from the definition of M_5 . The statement (b) follows from Lemmas 2.1 (b), 2.6, 2.9 (b), and from the definition of M_5 . Lemma 2.9 (c,d,e,f) implies Properties (d), (e), (f) and (g). □

3 Defining relations and derivation trapezia

3.1 Embedding homomorphism

Now we define an embedding of S in a finitely presented monoid H . Let A be a finite generator set of S , and let the machine M_5 be given by Lemma 2.10. We have $M_5 = \langle A_l, Y_l \sqcup Y_r, Q, \Theta, q_1 \rangle$, where $A_l \subset Y_l$ is the input alphabet which is the copy of A , $Y_l \sqcup Y_r$ is the tape alphabet (with left and right parts), Q is the set of states of M_5 , Θ is a set of commands, and $q_1 \in Q$ is the start state.

The set of generators of the monoid H is $A_H = A \sqcup Y_l \sqcup Y_r \sqcup Q \sqcup \{\alpha, \omega, p\}$ where α and ω are the endmarker symbols of M_5 , and p is one more generator. The set of defining relations of H is

$$R_H = \{V' = V \text{ for every command } V \rightarrow V' \text{ of } M_5\} \cup \quad (3.4)$$

$$\{pa = a_l p \text{ for every } a \in A \text{ and for its copy } a_l \in A_l\} \cup \{\alpha p = 1, p = q_1 \omega\} \quad (3.5)$$

Lemma 3.1. The identity map on the generator set A of S extends to a homomorphism $\phi : S \rightarrow H$. If S has 1 in the signature, then ϕ is a monoid homomorphism (i.e. $\phi(1) = 1$).

Proof. Assume that $u =_S v$. We must prove that $u =_H v$.

By Lemma 2.10 (a), there is an input-input computation C of M_5 starting with $\alpha u_l q_1 \omega$ and ending with $\alpha v_l q_1 \omega$, where u_l and v_l are the copies of u and v in the input alphabet A_l of M_5 . Since the relations $V = V'$ are included in R_H for all the commands $V \rightarrow V'$ of M_5 , all configurations of C are equal in H , in particular, $\alpha u_l q_1 \omega =_H \alpha v_l q_1 \omega$. Using the relation $q_1 \omega = p$, we obtain $\alpha u_l p =_H \alpha v_l p$. Now applying relations of the form $a_l p = pa$, we have $\alpha p u =_H \alpha p v$. Finally, $u =_H v$ since $\alpha p = 1$ by the definition of H . □

We will prove in Lemma 3.14 that ϕ is an injective homomorphism.

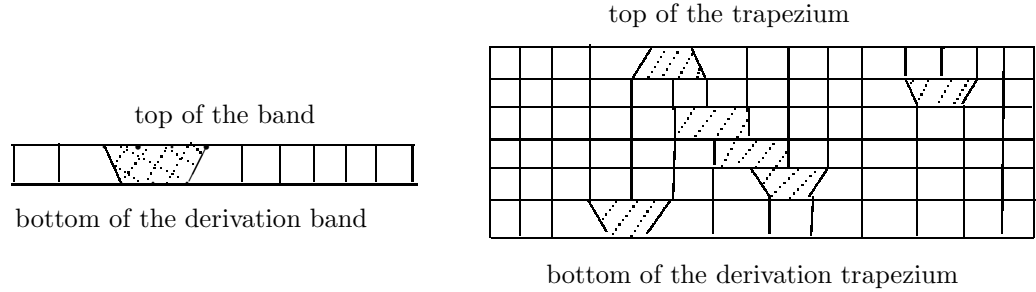
Remark 3.2. It follows from the proof of Lemma 3.1 and from Lemma 2.10(b) that for two equal in S words u and v of length at most n , we have $space_H(u, v) \leq S'_5(n) + 3$.

3.2 Derivation trapezia

Assume that $\mathcal{S} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ is a semigroup or monoid presentation. Then every derivation over this presentation has a visual geometric interpretation in terms of finite connected planar graphs. For group presentations, these graphs are called van Kampen diagrams (see [17]), and semigroup diagrams were introduced by Kashintsev (see [16] and [24]). Below we use a modified approach. Our diagrams uniquely restore derivations, which is preferable when one compares derivations with the computations of a TM. We call such diagrams *derivation trapezia* since they look similar to trapezia constructed from bands and associated with group computations (see [25], [26], [21], [4], etc.)

Every *cell* π is a trapezium in Euclidean plane with horizontal top and bottom. The top and the bottom of a *trivial* cell are labeled by the same letter from \mathcal{A} . In the *relation* cell π corresponding to a nontrivial relation $u = v$ from \mathcal{R} , the bottom is labeled by the word u and the top is labeled by v . This means that the bottom (the top) is divided into $|u|$ (resp., $|v|$) subsegments of nonzero length, each of the subsegments has a label from \mathcal{A} , and one reads the word u (the word v) on the bottom (on the top) from left to right. The sides of the trapezium π have no labels. Note that π can be a triangle if $|u| = 0$ or $|v| = 0$; but we will not include the trivial relations of the form $1 = 1$ in \mathcal{R} . Also we assume that \mathcal{R} is symmetric, i.e., a relation $u = v$ belongs to \mathcal{R} iff $v = u$ is in \mathcal{R} .

For every transition $w'uw'' \rightarrow w'vw''$, where $u = v$ is a defining relation from \mathcal{R} , we construct a *derivation band* as follows. We draw a horizontal parallel paths in the plane, the top and the bottom path directed from left to right. The bottom path (the top path) has $|w'uw''|$ (resp., $|w'vw''|$) edges of nonzero length, each of them is labeled by a letter from \mathcal{A} so that the label of the bottom (the top) path is $w'uw''$ (resp. $w'vw''$). Then we connect the initial (the terminal) vertex of the subsegment labeled by u in the bottom with, respectively, the initial (the terminal) vertex of the subsegment labeled by v in the top. This gives us the relation cell corresponding to the relation $u = v$. Finally, we connect the corresponding vertices of the top and the bottom to obtain $|w'| + |w''|$ trivial cells of the constructed derivation band. The left-most and the right-most connecting segments are, respectively, the left and the right *sides* of the derivation band.



It is obvious that every band with at most one transition cell corresponds to an elementary transition $w \rightarrow w'$. (We allow trivial transitions $w \rightarrow w$. The corresponding bands have only trivial cells. Note that the band corresponding to the transition $1 \rightarrow 1$ has no cells, but it has unlabeled side edges.)

Let $w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_t$ be a derivation over \mathcal{S} . Then the *derivation trapezium* Δ of height t corresponding to this derivation is composed of t derivation bands, where the bottom of the derivation band \mathcal{T}_{i+1} corresponding to the transition $w_i \rightarrow w_{i+1}$ coincides with the top of the derivation band \mathcal{T}_i corresponding to $w_{i-1} \rightarrow w_i$ ($i = 1, \dots, t-1$). Thus the label of the bottom (of the top) of Δ is w_0 (resp., w_t). The left (the right) sides of the derivation bands \mathcal{T}_i -s form the *left side* (the *right side*) of Δ .

We see that every derivation produces a derivation trapezium, and vice versa, every trapezium composed of derivation bands as above, is a derivation trapezium for some derivation (which may admit trivial transitions). Every horizontal edge of a derivation trapezium is labeled, and every vertical one (i.e., connecting the top and the bottom of a derivation band) is unlabeled.

A path is *vertical* if every its edge is vertical and different edges cross different derivation bands.

We call a derivation trapezium Δ *indivisible* if the only vertical paths connecting the top and the bottom of Δ are the left and the right sides of Δ .

Remark 3.3. If Δ corresponds to a derivation $w_0 \rightarrow \cdots \rightarrow w_t$, and it is divisible, then Δ is a union of two derivation trapezia of the same height: Δ_1 and Δ_2 , where $w_0(1)$ and $w_t(1)$ are bottom and top labels of Δ_1 , $w_0(2)$ and $w_t(2)$ are bottom and top labels of Δ_2 , $w_0 = w_0(1)w_0(2)$, $w_t = w_t(1)w_t(2)$. The derivation trapezium Δ_1 (resp., Δ_2) corresponds to a derivation $w_0(1) \rightarrow \cdots \rightarrow w_t(1)$ (to $w_0(2) \rightarrow \cdots \rightarrow w_t(2)$), where some transitions may be trivial. This observation reduces the study of the properties of derivation trapezia to indivisible ones.

Also we can apply the following *time separation trick* to the divisible derivation trapezium Δ . Since for every $i \leq t$, either transition $w_{i-1}(1) \rightarrow w_i(1)$ or the transition $w_{i-1}(2) \rightarrow w_i(2)$ is trivial (does not change the word), one may switch the order of the corresponding transitions in the derivation $w_0 \rightarrow \cdots \rightarrow w_t$ as follows:

$$w_0(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_t(2), \quad (3.6)$$

where the length of this derivation is t (not $2t$) since some trivial transitions are now omitted in first and in the second parts of (3.6). Thus, the corresponding derivation trapezium Δ' has the same height as Δ and the same bottom and top labels. The derivation subtrapezia Δ'_1 and Δ'_2 of Δ' are *time separated*: the derivation bands corresponding to the nontrivial transitions in Δ_2 follow after the transitions bands corresponding to the nontrivial transitions in Δ_1 (or vice versa). Note that the space of Derivation (3.6) can be greater than the space of the original derivation.

3.3 Vertical bands in trapezia over the monoid H

derivation bands are horizontal. Now we consider derivation trapezia over the presentation $H = \langle A_H \mid R_H \rangle$ and define vertical bands, namely q -bands, α -bands, ω -bands and a -bands.

By definition, a q -letter is a letter from $Q \cup \{p\}$. A q -edge is an edge labeled by a q -letter, a q -cell is a cell, having a q -edge in its top or bottom. (So every p -cell, i.e., having a boundary edge labeled by p , is also a q -cell.) A q -band of length n in a derivation trapezia Δ is a sequence of q -cells π_1, \dots, π_n such that if a cell π_i belongs to a derivation band \mathcal{T}_j , then π_{i+1} belongs to \mathcal{T}_{j+1} and these two cells share a q -edge ($i = 1, \dots, n-1$).

A q -band \mathcal{C} is called *maximal* if it is not contained in a longer q -band. It follows from the list of defining relations of H that the first cell π_1 (the last cell π_n) of \mathcal{C} either shares a q -edge with the bottom of Δ (resp. with the top of Δ) or it is an αp -cell, i.e., a cell corresponding to the relation $1 = \alpha p$ (to the relation $\alpha p = 1$, resp.).

Similarly one defines α - and ω -edges, α -bands and ω -bands. The properties of the first and the last cells of maximal α -bands are similar to the properties of the maximal q -bands mentioned above. The first cell π_1 (the last cell π_n) of a maximal ω -band either shares an ω -edge with the bottom of Δ (resp. with the top of Δ) or it is a $q_1\omega$ -cell, i.e., a cell corresponding to the relation $p = q_1\omega$ (to the relation $q_1\omega = p$, resp.).

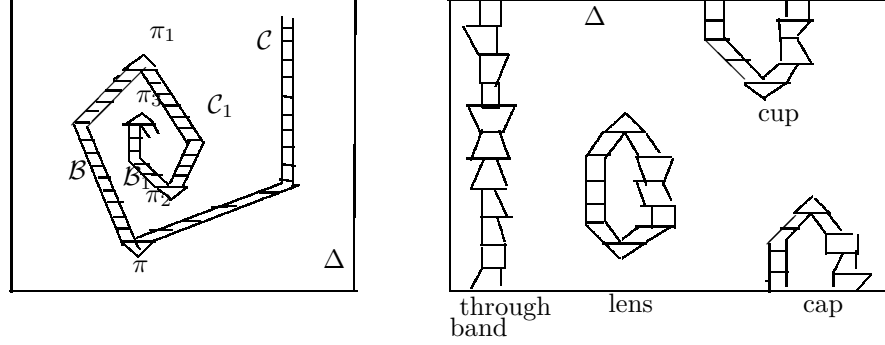
An a -edge is an edge labeled by a letter of the alphabet $A \cup Y_l \cup Y_r$, which, by definition, consists of a -letters. By definition, an a -band consists of trivial a -cells π_i -s with one a -letter written on the bottom and with the same letter labeling the top of π_i -s. A maximal a -band must start (end) either on the bottom (resp., top) of Δ or on the boundary of a q -cell having an a -letter in its top (resp., bottom) label.

The above definitions imply that a cell cannot belong to two different maximal q -bands (resp., α -bands, ω -bands, a -bands). If an α -band \mathcal{B} and a q -band \mathcal{C} start with the same αp -cell π , then any derivation band going from left to right and crossing both \mathcal{B} and \mathcal{C} , must first cross \mathcal{B} and then it crosses \mathcal{C} (or it crosses a cell shared by \mathcal{B} and \mathcal{C}). So we may say that the band \mathcal{C} is disposed from the right of \mathcal{B} . This simple observation leads to

Lemma 3.4. *Assume that an α -band \mathcal{B} and a q -band \mathcal{C} of a derivation trapezium Δ start or end with the same αp -cell π . Then either they end (resp., start) with the same αp -cell π' or they both reach the top (resp., bottom) of Δ .*

Proof. Proving by contradiction, we assume that these two bands start with π , and band \mathcal{B} is not longer than \mathcal{C} . (The other cases are similar.) Then \mathcal{B} is disposed from the left of \mathcal{C} , and the last cell of \mathcal{B} is an αp -cell π_1 . Then some q -band \mathcal{C}_1 must also terminate at π_1 , and \mathcal{C}_1 has to be placed from the right of \mathcal{B} and from the left of \mathcal{C} . Since it cannot start with π , \mathcal{C}_1 has to start with an αp -cell π_2 belonging to a

derivation band situated above the derivation band containing the cell π . Similarly, an α -band \mathcal{B}_1 must start with π_2 , it is disposed from the left of \mathcal{C}_1 and from the right of \mathcal{B} , and its last cell is an αp -cell $\pi_3 \neq \pi_1$. Reasoning this way, we can get arbitrarily many cells in Δ , a contradiction. \square



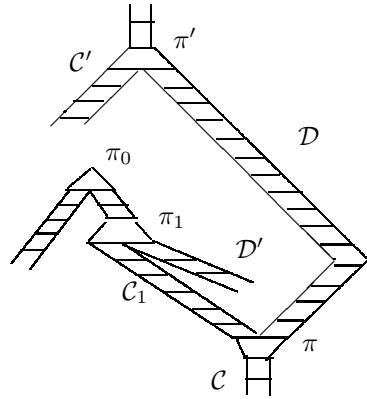
Lemma 3.4 implies that there can exist maximal α - and q -bands of three types in a trapezium Δ :

- (1) The bands connecting an α -edge (or a q -edge) of the bottom of Δ with an α -edge (or a q -edge) of the top. We call such bands *through bands*.
- (2) Pairs formed by an α -band and a q -band, sharing their the first and the last cells. We call such a pair an αq -*lens*.
- (3) Pairs formed by an α -band and a q -band, sharing the first (the last) cell and terminating (resp., starting) on the α - and q -edges of the top (resp., bottom) of Δ . We say that such a pair form an αq -*cup* (resp., αq -*cap*).

Lemma 3.5. *Let π and π' be, resp., the first cell and the last cell of a maximal ω -band \mathcal{D} . Then*

- (a) *π and π' cannot belong to different maximal q -bands.*
- (b) *If π belongs to the maximal q -band \mathcal{C} of a αq -lens, then π' also belongs to \mathcal{C} .*
- (c) *If an αq -cap (or cup) Γ surrounds no smaller caps (resp. cups), then Γ surrounds no ω -bands starting on the bottom (resp., on the top) of Δ .*

Proof. (a) Arguing by contradiction we assume that \mathcal{D} is a shortest counter-example. It starts on some maximal q -band \mathcal{C} and ends on a maximal q -band \mathcal{C}' . Since \mathcal{D} is situated from the right of both \mathcal{C} and \mathcal{C}' , and these two q -bands do not cross, either \mathcal{C} does not reach the top of the trapezium Δ or \mathcal{C}' does not start on the bottom of Δ . Choosing the former case, we deduce that \mathcal{C} ends with a αp -cell π_0 , and the subband \mathcal{C}_1 of \mathcal{C} with the first cell π and the last one π_0 is shorter than \mathcal{D} .



Since π has a top edge labeled by q_1 and π_0 has a bottom edge labeled by p , there must be a cell in \mathcal{C}_1 which corresponds to the relation $q_1\omega = p$. Moreover, the number of such cells in $\mathcal{C}_1 \setminus \pi$ must be greater

than the number of cells corresponding to the transition $p \rightarrow q_1\omega$. Therefore there is a cell in \mathcal{C}_1 , say π_1 , such that a maximal ω -band \mathcal{D}' ends with π_1 but it does not start on \mathcal{C}_1 . This ω -band \mathcal{D}' is situated from the right of \mathcal{C} and from the left of \mathcal{D} . Since the bands \mathcal{C} and \mathcal{D} have the common cell π , the band \mathcal{D}' is shorter than \mathcal{C}_1 , and consequently, it is shorter than \mathcal{D} .

We come to a contradiction with the choice of \mathcal{D} , and Claim (a) is proved.

(b) The assumption that \mathcal{D} starts on \mathcal{C} and ends on the top of Δ provides us, as in the proof of (a), with an ω -band \mathcal{D}' connecting two different maximal q -bands. Thus Property (b) is proved by contradiction.

(c) Follows from (b) since an ω -band cannot start and end on the bottom (resp., on the top) of Δ . \square

3.4 Minimal trapezia

If a q -band \mathcal{C} of a derivation trapezium Δ over H has k $q_1\omega$ -cells, then we say that \mathcal{C} has type k . Suppose Δ has τ_i through q -bands of type i , σ_i maximal q -bands of type i in the αq -caps and αq -cups, and ρ_i maximal q -bands of type i in the αq -lenses, $i = 0, \dots, k$, and Δ has no q -bands of types $> k$. Then we say that Δ is a trapezium of type $\tau(\Delta) = (\tau_0, \sigma_0, \rho_0, \dots, \tau_k, \sigma_k, \rho_k, 0, 0, 0 \dots)$.

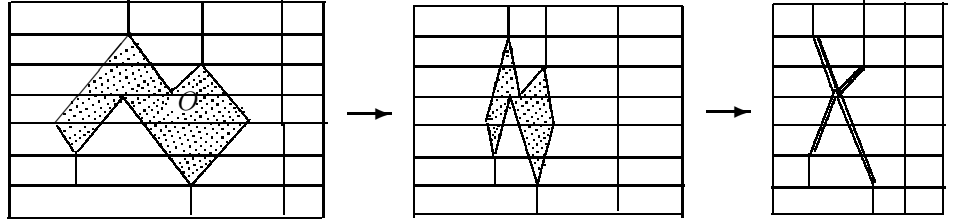
Assume that $\tau(\Delta') = (\tau'_0, \sigma'_0, \rho'_0, \dots, \tau'_k, \sigma'_k, \rho'_k, 0, 0, 0 \dots)$. Then by definition $\tau(\Delta) > \tau(\Delta')$ if there is l such that $\tau_l > \tau'_l$ or $\tau_l = \tau'_l$ and $\sigma_l > \sigma'_l$, or $\tau_l = \tau'_l$ and $\sigma_l = \sigma'_l$, but $\rho_l > \rho'_l$, and $\tau_m = \tau'_m$, $\sigma_m = \sigma'_m$, $\rho_m = \rho'_m$ for every $m \geq l$.

Clearly, the defined order on derivation trapezia over H satisfies the descending chain condition, and so there is a trapezium having the smallest type among all trapezia with the same bottom and top labels. Such a derivation trapezium is called a *minimal trapezium*.

Remark 3.6. It is easy to see that the time separation trick from Remark 3.3 preserves the numbers of αp -cells and $q_1\omega$ -cells in every maximal q -band, and so it does not change the types of maximal q -bands. Therefore it preserves the minimality of a trapezium. The same is true if one rebuilds two derivation band of a derivation trapezium replacing a subderivation $w \rightarrow w \rightarrow w'$ by $w \rightarrow w' \rightarrow w'$ or vice versa.

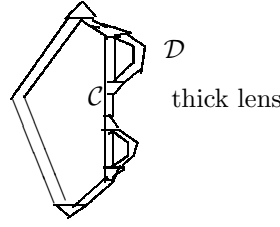
Lemma 3.7. Assume that \mathbf{p} is a simple closed path in a minimal trapezium Δ , and every edge of \mathbf{p} is unlabeled. Then the closed region O of Δ bounded by \mathbf{p} contains no q -edges.

Proof. Let us shrink to a point every labeled (horizontal) edge which is inside O . If after this surgery some unlabeled (vertical) edges connect the same vertices, we identify such edges.



It is clear that we replace every derivation band of Δ by a derivation band of the obtained trapezium Δ' (but the cells belonging to O are removed), Δ' has the same top and bottom labels as Δ , and $\tau(\Delta') < \tau(\Delta)$ if O has at least one q -edge (and therefore contains a maximal q -band). Since Δ is a minimal trapezium, the lemma is proved. \square

If a maximal ω -band \mathcal{D} starts on the right side of the maximal q -band \mathcal{C} of a αq -lens E then \mathcal{D} also terminates on \mathcal{C} by Lemma 3.5 (b). Let us attach all such maximal ω -bands to \mathcal{C} and call the obtained figure Γ a *thick lens*.



Lemma 3.8. *Every edge of the outer boundary component \mathbf{x} of a thick lens Γ is either unlabeled or labeled by a letter from the alphabet A .*

Proof. Every edge of an α -cell (of an ω -cell) of Γ lying on \mathbf{x} is unlabeled since α (resp., ω) can occur only as the left-most (resp., the right-most) letter in the relator words of H . It follows from the definitions of bands and Γ that the edges of αp -cells and of $q_1\omega$ -cells belonging to \mathbf{x} are unlabeled too.

Since the cells of \mathcal{C} corresponding to the relation $p = q_1\omega$ and to $q_1\omega = p$ alternate in its maximal q -band \mathcal{C} of Γ , every maximal ω -band of Γ must start with a cell of \mathcal{C} corresponding to $p = q_1\omega$ and end on the next $q_1\omega$ -cell of \mathcal{C} corresponding to $q_1\omega = p$. Hence the only cells of \mathcal{C} having edges in \mathbf{x} are p -cells, and labeled edges of \mathbf{x} are their a -edges on the right side of \mathcal{C} . So they are labeled by letters from A (see Relations (3.5)). \square

We say that a closed region O in a derivation trapezium is *generated* by a thick lens Γ (or by the αq -lens E defining Γ) if (1) O contains Γ ; (2) if O contains an edge e of a cell π and e is labeled by a letter from A , then O contains π ; (3) if O contains an edge from the outer boundary of some thick lens Γ' , then O contains Γ' (4) O is minimal with respect to (1)–(3).

Lemma 3.9. *Let a region O of a derivation trapezium is generated by a αq -lens E or by a thick lens Γ . Then every edge in the outer boundary component of O is either unlabeled or has a label from A .*

Proof. By the definition of O , it constructed from several thick lenses and several maximal a -bands which start/terminate on the thick lenses and correspond to a -letters from A . So Lemma 3.8 completes the proof. \square

Lemma 3.10. *Let Δ be a minimal trapezium over H . Then*

- (a) *An αq -lens Γ of Δ encloses no other αq -lenses and no ω -cells.*
- (b) *Let Δ have a through q -band \mathcal{C} , and assume that the top and bottom edges of Δ from the left of \mathcal{C} are labeled by letters from $\{\{\alpha\} \cup Y_i\}$. Then there are no αq -lenses and no ω -cells from the left of \mathcal{C} .*
- (c) *Assume that an αq -cap (or cup) Γ of Δ encloses a αq -lens E . Then the closed region O generated by E does not share any labeled edge with Γ .*

Proof. (a) Assume that an αq -lens E is enclosed in Γ , and there are no bigger αq -lenses enclosed in Γ and surrounding E . Note that the region O generated by the αq -lens E is also enclosed in Γ . By Lemma 3.9, every labeled edge e of the outer boundary component \mathbf{p} of O must be connected by a maximal a -band \mathcal{A} (of length ≥ 0) with an a -edge f on the left side of the maximal q -band \mathcal{C} of Γ . However f is labeled by a letter from Y_i while e is labeled by a letter from A . This contradicts to the condition $Y_i \cap A = \emptyset$, and so \mathbf{p} has no labeled edges.

Then Lemma 3.7 gives another contradiction since O contains q -edges of the αq -lens E . Hence our assumption false, and Γ surrounds no αq -lenses.

The second assertion of (a) is also true since an ω -band enclosed in Γ cannot start/end on \mathcal{C} .

(b) The same proof as for (a), but now Γ is the part of Δ from the left of \mathcal{C} .

(c) Follows from Lemma 3.9. \square

Lemma 3.11. (a) Assume that an a -band \mathcal{A} starts and ends on a q -band \mathcal{C} of a minimal trapezium, and \mathcal{C} has no edges labeled by p . Then \mathcal{A} and \mathcal{C} surrounds no αq -lenses and no ω -cells.

(b) Let an ω -band \mathcal{D} start and end on a q -band \mathcal{C} of a minimal trapezium Δ . Then \mathcal{C} and \mathcal{D} surround no ω -, α -, or q -cells.

Proof. (a) It follows from the assumption of the lemma that the a -band \mathcal{A} corresponds to a letter from Y_l (from Y_r) if it is disposed from the left (resp., from the right) of \mathcal{C} . Now the assumption that \mathcal{A} and \mathcal{C} surround an αq -lens E gives a contradiction as in Lemma 3.10. The second claim is obvious since \mathcal{C} has no p -edges, and so no ω -band can start on the band \mathcal{C} .

(b) Let \mathcal{D} start with a $q_1\omega$ -cell π of \mathcal{C} . Then π corresponds to the relation $p = q_1\omega$. Therefore the next $q_1\omega$ -cell π' of \mathcal{C} must correspond to the relation $q_1\omega = p$, and some maximal ω -band \mathcal{D}' terminates at π' . Since by Lemma 3.5, \mathcal{D}' must also start on \mathcal{C} , and different maximal ω -bands cannot cross each other, we conclude that $\mathcal{D}' = \mathcal{D}$. In other words, if the closed region Γ bounded by \mathcal{C} and \mathcal{D} (where the cells from \mathcal{C} and \mathcal{D} do not belong to Γ) encloses a maximal ω -band, then such a band must connect two cells of an αq -lens enclosed in Γ . If Γ contains α - or q -cells, then Γ contains αq -lenses as well. But such an assumption leads to a contradiction exactly as in the ‘right’ version of Part (a). \square

3.5 Types of q -bands in minimal trapezia

A derivation trapezium Δ will be called a *machine trapezium* if the top or the bottom label w of Δ is a configuration of the machine M_5 and every nontrivial cell corresponds to one of the machine relation (3.4) (i.e., there are no cells corresponding to the auxiliary relations (3.5)).

Let w and w' be the bottom and the top labels of a derivation band of a machine trapezium Δ . Then it follows by the induction on the height of Δ that either $w' = w$ or this band correspond to a transition $w \rightarrow w'$ of M_5 . Therefore both the bottom and the top labels of Δ are configurations of M_5 , and they can be connected by a computation of M_5 .

The definition of *peeled machine trapezium* is similar but now w can be a configuration of M_5 without one of the endmarkers α or ω , or without both. Hence the bottom and top labels of a peeled machine trapezium plus the additional letter α in the beginning or/and the letter ω at the end of them are connected by a computation of M_5 without commands involving α or ω , or both, resp.

Lemma 3.12. Assume that Δ be a minimal trapezium over H .

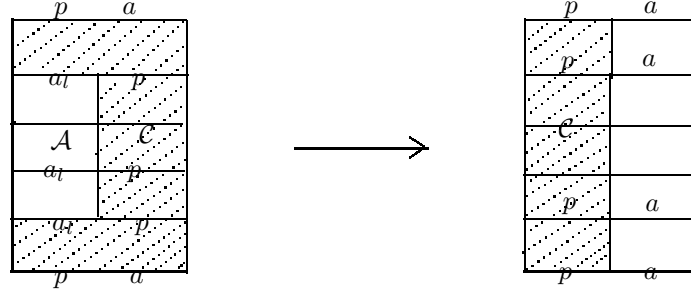
(a) Let Γ be an αq -lens in Δ formed by an α -band \mathcal{B} and q -band \mathcal{C} . Then the type of \mathcal{C} is 2.

(b) Assume that Δ has a through q -band \mathcal{C} and a through α -band \mathcal{B} from the left of \mathcal{C} . Then there is no horizontal path \mathbf{x} starting with an α -edge of \mathcal{B} , ending with an p -edge of \mathcal{C} , and having label $\alpha U p$, where U is a word in the alphabet A_l .

(c) Assume that Δ has an αq -cap or an αq -cup Γ with maximal α -band \mathcal{B} and maximal q -band \mathcal{C} . Also assume that there are neither αq -lenses nor αq -caps/cups enclosed in Γ . Then the type of \mathcal{C} is at most 1.

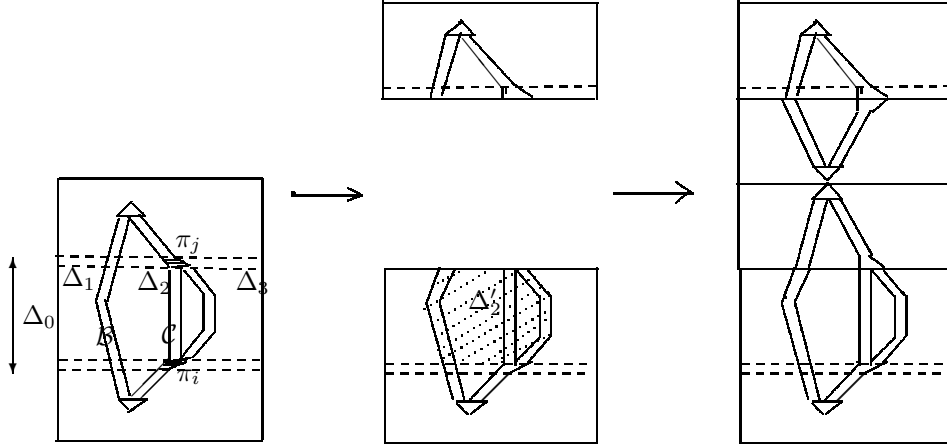
Proof. (a) Denote by π_1 and π_m the first and the last cells of \mathcal{C} . They correspond to the relations $1 = \alpha p$ and $\alpha p = 1$, resp. Therefore \mathcal{C} has equal number of cells corresponding to the relation $p = q_1\omega$ and to $q_1\omega = p$, in particular, the type t of \mathcal{C} is even.

Case 1. Assume that $t = 0$. Then each of the cells π_2, \dots, π_{m-1} is either trivial or corresponds to a relation $pa = a_ip$ (or to $a_ip = pa$). Note that by Lemma 3.10, there are neither αq -lenses nor ω -cells enclosed in Γ . Hence every cell between \mathcal{B} and \mathcal{C} is a trivial a -cell and it belongs to an a -band starting and ending on \mathcal{C} and corresponding to a letter from A_l . A right-most a -band \mathcal{A} enclosed in Γ and a part of \mathcal{C} form a derivation (sub)trapezium with top and bottom labels equal to pa ($a \in A$). Therefore one can replace this subtrapezium by a trapezium having only trivial cells.



This surgery reduces the number of nontrivial cells in the q -band \mathcal{C} . Finally, we will have an αq -lens with unlabeled (outer) boundary. Hence the αq -lens can be removed from Δ by Lemma 3.7. Since Δ is a minimal trapezium, the case $t = 0$ is not possible.

Case 2. Assume now that $t \geq 4$, and let π_i and π_j be the first cells of \mathcal{C} corresponding to the relations $p = q_1\omega$ and $q_1\omega = p$, respectively. Let Δ_0 be the trapezium formed by the derivation bands $\mathcal{T}_k, \dots, \mathcal{T}_{k+j-i}$ of Δ containing π_i, \dots, π_j , resp. There are two vertical paths dividing Δ_0 : The left side of \mathcal{B} and the right side of the ω -band starting with π_i and ending with π_j (see Lemma 3.5). These paths divide Δ_0 into 3 subtrapezia $\Delta_1, \Delta_2, \Delta_3$ (from left to right).



Applying the time separation trick (see Remarks 3.3 and 3.6) with possible decrease of the length of \mathcal{C} , we may assume that the derivation corresponding to Δ_2 has no trivial transitions, in particular, the middle part of this derivation of length $j - i - 1$ has this property too. It corresponds to the trapezium Δ'_2 obtained from Δ_2 by the deleting of the first and the last derivation bands.

Observe that by Lemma 3.10, Δ'_2 has no cells corresponding to the auxiliary relations (3.5), and the bottom label of it is of the form $\alpha u q_1 \omega$ for some word u in the tape alphabet of M_5 . In fact, u is a word in A_l since by the definition of π_i , an a -band ending on the bottom of Δ'_2 starts on a cell of \mathcal{C} having an edge labeled by p . Therefore $\alpha u q_1 \omega$ is a configuration of M_5 , and moreover, it is an input configuration, and Δ'_2 is a machine trapezium.

The derivation trapezium Δ'_2 corresponds to a computation C of M_5 . By Lemma 2.10 (g), the computation C must be an input-input computation, since it ends with a configuration $\alpha u' q_1 \omega$.

Thus using (3.5) we can construct the following derivation D starting with the bottom label of \mathcal{T}_{k+j-i} (this word contains the subword $\alpha u' q_1 \omega$):

$$(\dots \alpha u' q_1 \omega \dots) \rightarrow (\dots \alpha u' p \dots) \rightarrow \dots \rightarrow (\dots \alpha p u'' \dots) \rightarrow (\dots u'' \dots),$$

where u'' is the copy of u' in the alphabet A . This property makes possible the following surgery with Δ . We cut Δ along the bottom path of \mathcal{T}_{k+j-i} , and insert mutually mirror trapezia corresponding to the derivation D and to its inverse. Since D removes the α - and q -letters in the distinguished subword, this surgery replaces the αq -lens Γ by an αq -lens with maximal q -band of type 2 and an αq -lens with maximal q -band of type $t - 2$. Since all maximal q -bands of Δ , except for \mathcal{C} are untouched by this surgery (more

precisely, we added several trivial cells to some of them), the obtained trapezium has smaller type than Δ , a contradiction.

(b) Proving by contradiction, we may assume that the top or bottom of some derivation band contains the subpath \mathbf{x} . So its label is of the form $\dots \alpha U p \dots$. Again due to Relations (3.5), $\alpha U p = U'$, where U' is the copy of U in A . Hence one can use the same trick as in (a) and replace the through bands \mathcal{B} and \mathcal{C} by a cap and a cup, and the sum of types of their q -bands is equal to the type of \mathcal{C} , contrary to the minimality of Δ .

(c) We may assume that Γ is a cup. It follows from the assumptions of the lemma that there are no cells corresponding to Relations (3.5) surrounded by Γ and the top of Δ . If the type of \mathcal{C} is at least two, then, as in Case (a), one can consider two $q_1\omega$ -cells π_i and π_j in \mathcal{C} and then using similar surgery, replace the cup Γ by a cup of smaller type and an αq -lens. So our assumption leads to a contradiction with the minimality of Δ .

The lemma is proved. \square

Remark 3.13. (a) By Lemma 3.12, the maximal q -band \mathcal{C} of an αq -lens E in a minimal trapezium Δ has exactly two $q_1\omega$ -cells, say, π_i and π_j . By Lemma 3.5 (a), these two cells are connected (from the right of \mathcal{C}) by a maximal ω -band \mathcal{D} . We obtain a thick lens Γ by adding \mathcal{D} to Γ . The cells of \mathcal{C} under π_i and above π_j (including π_i and π_j) correspond to the auxiliary relations (3.5). So the edges of the outer boundary of the thick lens are either unlabeled or labeled by letters from A .

(b) One can argue as in Case 2 of Lemma 3.12 (though $t = 2$ now) and obtain the subtrapezium Δ_0 and its parts $\Delta'_2 \subset \Delta_2$. As in the proof of Lemma 3.12, we may assume that Δ'_2 is a machine trapezium, and every derivation band of it corresponds to a (non-trivial) command of an input-input computation of M_5 , and so the top and the bottom of Δ_0 have labels of the form $w'u_lw''$ and $w'v_lw''$, where u_l and v_l are words in the alphabet A_l , and the copies u and v of these words in the alphabet A are equal in S by Lemma 2.10 (a). We will call $\Delta'_2 = M(\Gamma)$ the *machine part* of the thick lens Γ ; $\Delta_2 = \bar{M}(\Gamma)$ is the *augmented machine part* of Γ . (It worths to note that it contains nontrivial $q_1\omega$ -cells in the first and in the last derivation bands.)

Lemma 3.14. *The homomorphism $\phi : S \rightarrow H$ defined in Lemma 2.10 is injective.*

Proof. Let w and w' be two words in generators of S , i.e., in the alphabet A . Assuming that $w =_H w'$, we must prove that $w =_S w'$. So we have a derivation $w = w_0 \rightarrow \dots \rightarrow w_t = w'$ over H and denote by Δ the corresponding minimal derivation trapezium. Since the boundary labels w and w' of Δ have neither α - nor q -letters, all maximal α - and q -bands (if any) are paired in some αq -lenses E_1, \dots, E_k , and neither of the corresponding thick lenses $\Gamma_1, \dots, \Gamma_k$ is enclosed in another one by Lemmas 3.10 and 3.11 (b).

We may assume that the first derivation band of Δ containing a $q_1\omega$ -cell from $\cup_{l=1}^k E_l$ (it exists by Lemma 3.12 if $k > 0$) does contain a cell from E_1 , and so it does not contain other $q_1\omega$ -cells. Let $\bar{M}(\Gamma_1)$ be the augmented machine part of the lens Γ_1 (see Remark 3.13). As in Remark 3.13, we may use the time separation trick, and have each of the lowest $q_1\omega$ -cells of $\Gamma_2 \dots \Gamma_k$ disposed in the derivation bands of Δ with higher numbers than the derivation bands containing any of the cells of Γ_1 . Therefore the time separation trick can now be applied to Γ_2 . This reconstruction does not touch $\bar{M}(\Gamma_1)$ and creates $\bar{M}(\Gamma_2)$ with cells disposed above the derivation bands of Δ crossing $\bar{M}(\Gamma_1)$. Finally, we replace Δ by a minimal trapezium with the same top and bottom labels, where the augmented machine part $\bar{M}(\Gamma_i)$ lies above $\bar{M}(\Gamma_{i-1})$ for $i = 2, \dots, k$.

We will keep the same notation Δ for the obtained trapezium. In every word w_i of the derivation $w_0 \rightarrow \dots \rightarrow w_t$ corresponding to Δ , we delete all letters which do not belong to $A \cup A_l$, replace every letter from A_l by its copy from A and denote the obtained word from A^* by $\psi(w_i) = W_i$.

By Remark 3.13, $W_r =_S W_s$ if w_r and w_s include, resp., the top and the bottom labels of some $M(\Gamma_i)$. If E is a trapezium formed by the derivation bands of Δ situated between $M(\Gamma_{i-1})$ and $M(\Gamma_i)$ (or between the bottom (the top) of Δ and $M(\Gamma_1)$ (and $M(\Gamma_k)$)), and W_r and W_s are ψ -images of the top and the bottom labels of E , then $W_r = W_s$, because the derivation $w_s \rightarrow \dots \rightarrow w_r$ uses only the auxiliary relation (3.5).

Consequently, $W_0 =_S W_t$, and so $w_0 = W_0 =_S W_t = w_t$, as required. \square

Lemma 3.15. *Let Δ be a minimal derivation trapezium over H with the bottom label $w_0 = (\alpha)UpV$ and the top label $w_t = (\alpha)U'pV'$, where U, U' are words in A_l , V, V' are words in A , and α can be absent in*

both labels. We assume that Δ has a through q -band \mathcal{C} . Then using notation of Lemma 3.14, we have $\psi(w_0) =_S \psi(w_t)$.

Proof. Let $\Gamma_1, \dots, \Gamma_k$ be all the thick lenses of Δ ($k \geq 0$). By Lemma 3.10 (b) none of them is placed from the left of \mathcal{C} . If the type of \mathcal{C} is equal to $2l \geq 0$, then it has $2l$ $q_1\omega$ -cells, and using these cells one can define l (peeled) augmented machine trapezia $\bar{M}_1, \dots, \bar{M}_l$, where each \bar{M}_j is bounded from the left by a portion of a through α -band \mathcal{B} (or by the left side of Δ if there is no α -bands in Δ) and bounded from the right by an ω -band connecting some $q_1\omega$ -cells of \mathcal{C} . Note that by Lemma 3.11 (b) and the above observation, these (peeled) trapezia have no lenses.

Now one can apply the time separation trick to the system $\bar{M}_1, \dots, \bar{M}_l, \bar{M}(\Gamma_1), \dots, \bar{M}(\Gamma_k)$ as this was done for the augmented machine parts of $\Gamma_1, \dots, \Gamma_k$ in the proof of Lemma 3.14. So as there, we will have $\psi(w_0) =_S \psi(w_t)$, and the lemma is proved. \square

3.6 A-triangles in derivation trapezia

Assume that \mathbf{x} is a nontrivial subpath of the bottom (or of the top) of a trapezium Δ , and two vertical paths \mathbf{y} and \mathbf{z} start at \mathbf{x}_- (the original vertex of the path \mathbf{x}) and \mathbf{x}_+ (the terminal vertex of \mathbf{x}), resp. If $\mathbf{y}_+ = \mathbf{z}_+$, and there are no other common vertices of \mathbf{y} and \mathbf{z} , then we say that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ bound a triangle subtrapezium Δ_0 of Δ . (A triangle trapezium corresponds to a derivation ending or starting with the empty word 1.) If the *base* \mathbf{x} is labeled by a word in A , we say that Δ_0 is an *A-triangle*.

Lemma 3.16. (a) Assume that Γ is an αq -cap or an αq -cup in a minimal trapezium Δ , and there are no other αq -caps (resp., cups) enclosed in Γ . Assume that there is an αq -lens E enclosed in Γ . Then there is an *A-triangle* Δ_0 in Δ , containing E and enclosed in Γ .

(b) Let Γ be a triangle in a minimal trapezium Δ . Assume that there are no αq -caps or αq -cups but there is an αq -lens E enclosed in Γ . Then there is an *A-triangle* Δ_0 containing E and enclosed in Γ .

(c) Assume that a q -band \mathcal{C} and an ω -band \mathcal{D} start (or end) with the same $q_1\omega$ -cell and end (resp. start) on the top (resp., on the bottom) of Δ . If there are no αq -caps or αq -cups but there is an αq -lens surrounded by these two bands and by the top (by the bottom) of Δ , then Δ has an *A-triangle* containing E .

(d) Let \mathcal{C} be a through q -band without p -edges in a minimal indivisible trapezium Δ and \mathcal{D} a through ω -band from the right of \mathcal{C} . Suppose there are neither αq -caps, nor αq -cups, nor through bands between \mathcal{C} and \mathcal{D} , but there is an αq -lens E between them. Then there is an *A-triangle* Δ_0 containing E between \mathcal{C} and \mathcal{D} .

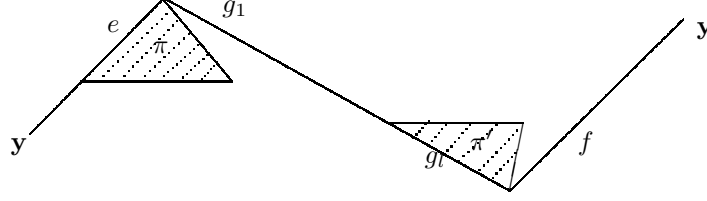
Proof. (a) We will assume that Γ is an αq -cap. Let O be the closed region generated by E (see Subsection 3.4). Note that by Lemma 3.10(c), no labeled edge of Γ belongs to O , and by Lemma 3.9 and the definition of O , every labeled edge of the outer boundary of O belongs to the bottom of Δ .

If O has no edges on the bottom of Δ , then the (outer) boundary of O has no labeled edges. This would contradict Lemma 3.7. So the region O is enclosed in a simple loop $\mathbf{x}\mathbf{p}$, where \mathbf{x} is the minimal subpath of the bottom of Δ containing all bottom edges belonging to O , and \mathbf{p} is an unlabeled path on the boundary of O .

We select a factorization $\mathbf{p} = \mathbf{z}\mathbf{y}^{-1}$, where the last edges of both \mathbf{y} and \mathbf{z} go upward, and consider two cases.

Case 1. Assume that both \mathbf{z} and \mathbf{y} are vertical paths. Then $\mathbf{x}, \mathbf{y}, \mathbf{z}$ bound a triangle trapezium Δ_0 . By Lemma 3.5 (b), \mathbf{x} has no ω -edges since the maximal ω -band starting on such an edge could not end anywhere. So every band starting on \mathbf{x} must reach an edge of O (or an edge of a closed region generated by another αq -lens). Therefore the label of this edge belongs to A by Lemma 3.9. Hence Δ_0 is an *A-triangle*.

Case 2. One of the paths \mathbf{y}, \mathbf{z} , say \mathbf{y} is not vertical. Then there is a subpath $eg_1 \dots g_l f$ ($l \geq 1$) in \mathbf{y} , where the edges e and f go upward, but all the edges g_1, \dots, g_l go downward.



Since both e^{-1} and g_1 are directed downward and they start from the same vertex, there must be a cell π in Γ corresponding to the relation $\alpha p = 1$ and having a common edge with the path eg_1 . Similarly, there is a cell π' , corresponding to the relation $1 = \alpha p$ and having an edge from the subpath $g_2 f$. Observe that if π belongs to O , then π' lies outside this region, and vice versa, since p is a part of the boundary of O . Let us assume that π does not belong to O . However the αp -cell π must belong to one of the αq -lenses enclosed in Γ . This contradicts the definition of the region O since π has an edge belonging to the boundary of O . Thus Case 2 is impossible, and Statement (a) is proved.

(b,c) The same proof as for (a) since two sides of the triangle are simply unlabeled now, and in Case (c), only a -band corresponding to the letters from Y_r can start on C .

(d) The proof is similar to that in (a) but there might happen that a segment of the boundary of the closed region O connects the top and the bottom of Δ . As in Case 2 above, this segment must be vertical. But this would imply that Δ is a divisible trapezium, a contradiction. \square

Let a minimal trapezium Δ have no A -triangles and have an αq -cup (or a cap) Γ satisfying the assumptions of Lemma 3.12 (c). We also assume that Δ has no cups (or caps) of smaller height than Γ (i.e., with maximal q -band shorter than C). We define the *base label* $b(\Gamma)$ of Γ as follows. If the type of the q -band C of Γ is 0 (and so all cells of C are p -cells), then $b(\Gamma)$ is just the word $\alpha W p$ we read on the top (or on the bottom) of Δ between the α -edge and the q -edge of Γ . If the type of C is 1, then C has one $q_1 \omega$ -cell, and so one maximal ω -band D starts on C from the right and, by Lemma 3.5 (a), ends on the top (or on the bottom) of Δ . Then $b(\Gamma)$ is the word $\alpha U q V \omega$ we read between the ends of B and D on the top/bottom of Δ .

Lemma 3.17. *Under the above restrictions, (1) if C has type 0, then W is a word in the alphabet A_l ; (2) if the type of C is 1, then the base label $b(\Gamma)$ is a reachable configuration of the machine M_5 .*

Proof. If the type of C is 0, then every maximal band starting on the segment labeled by W ends on a p -cell of C . This implies the first statement of the lemma. To proof the second one, we consider the derivation bands of Δ crossing D . They form a derivation trapezium Δ' . Let Δ'' be a subtrapezium of Δ' bounded from the left by the left side of B (which is vertical) and bounded from the right by the right side of D (which is vertical too).

The bottom of Δ'' has label of the form $\alpha W q_1 \omega$, where W is a word in A_l since the underlying part of the cup Γ has the q -band of type 0. The part of Δ'' between B and C has no auxiliary cells (corresponding to Relations (3.5)) because Γ surrounds no q -cells. There are no αq -cups/caps between C and D by the minimality of the height of Γ . Also there are no lenses there by Lemma 3.16 (d). Hence the part of Δ'' between C and D has no auxiliary cells too. Therefore Δ'' is a machine trapezium with bottom label $\alpha W q_1 \omega$, and so its top label $b(\Gamma)$ is reachable by M_5 , as required. \square

4 Indivisible trapezia and completion of proofs

4.1 Upper bounds for spaces of derivations

We will assume in Lemmas 4.1 - 4.7 that Δ is an indivisible minimal trapezium without caps, cups, and A -triangles. Let Δ correspond to a derivation $D : w_0 \rightarrow \dots \rightarrow w_t$ over H , and \mathbf{x}, \mathbf{y} are the top and the bottom of Δ , resp.

Lemma 4.1. Δ has at most one α -band (at most one ω -band) connecting \mathbf{x} and \mathbf{y} . If such a band exists, its left side (resp., right side) coincides with the left (resp., right) side of Δ .

Proof. The letter α (resp., ω) can occur in a defining relation $u = v$ of H only as the left-most (the right-most) letter of u or v . It follows that the left side of an α -band (the right side of an ω -band) connecting \mathbf{x} and \mathbf{y} is a vertical line. Since Δ is indivisible, this line must be equal to the left (to the right) side of Δ , and the statement of the lemma follows. \square

Lemma 4.2. If Δ has no through q -bands, then $\text{space}_H(w_0, w_t) \leq S_5(\max(|w_0|_a, |w_t|_a) + 3$.

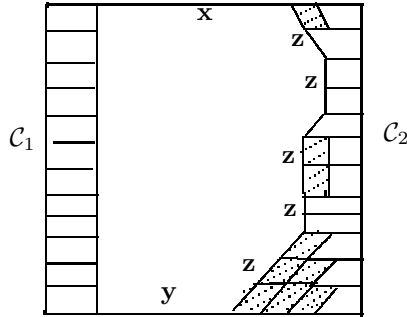
Proof. We may assume that $\text{space}(D) > 1$. Then there are no ω -bands connecting \mathbf{x} and \mathbf{y} since either the left or the right side of such a band would make Δ divisible. Similarly Δ has no through α -bands since any αq -cell of Δ must belong to a lens.

Therefore the top and bottom labels are words in $A \cup Y_l \cup Y_r$. By Remark 3.13 (a), every a -edge of the outer boundary of a thick lens is labeled by a letter from A . Hence every maximal a -band starting on \mathbf{y} with an edge labeled by a letter from $Y_l \cup Y_r$ consists of trivial cells only and ends on \mathbf{x} . This makes Δ divisible, a contradiction.

Thus the top and bottom labels are words in the alphabet A . By Lemma 3.14 we have $w_0 =_S w_t$, and Remark 3.2 completes the proof. \square

Lemma 4.3. Δ has at most one through q -band.

Proof. Assume that there are two through q -bands \mathcal{C}_1 and \mathcal{C}_2 , where \mathcal{C}_2 is from the right of \mathcal{C}_1 , and there are no through q -bands between them. Note that only maximal a -bands corresponding to the letters from Y_l can start on the left side of \mathcal{C}_2 . These a -band can end either on \mathcal{C}_2 or on the top/bottom of Δ . Hence there is a vertical path \mathbf{z} connecting \mathbf{x} and \mathbf{y} whose edges belong either to the left sides of some of these a -bands or to the left side of \mathcal{C}_2 . It follows that the derivation trapezium Δ is divisible by this vertical line, a contradiction. The lemma is proved. \square



Lemma 4.4. Assume that Δ has one through q -band \mathcal{C} . Then

- (1) each cell from the left of \mathcal{C} is a trivial a -cell corresponding to a letter from Y_l or it is an α -cell;
- (2) if Δ has no through ω -bands, and \mathcal{C} has no cells having an edge labeled by p , then every q -cell of Δ belongs to \mathcal{C} and all a -edges from the right of \mathcal{C} are labeled by letters from Y_r .

Proof. We will prove Statement (2) of the lemma. By the assumptions, no ω -bands can start/end on \mathcal{C} . Let \mathcal{A}_1, \dots be all the maximal a -bands starting on \mathcal{C} from the right. Since no cell of \mathcal{C} has a p -edge, all these bands correspond to letters from Y_r . Hence each of \mathcal{A}_i -s must end either on \mathcal{C} or on \mathbf{x} , or on \mathbf{y} (but cannot end on the outer boundary of a thick lens by Remark 3.13 (a)). By Lemmas 3.11 (a) and 3.16 (b), there are neither lenses nor ω -cells between any two of these a -bands or between some \mathcal{A}_i and \mathcal{C} .

There is a vertical line composed of the side edges of these a -bands and of \mathcal{C} . Since Δ is indivisible, this line coincides with the right side of Δ , and so every cell in and from the right of \mathcal{C} corresponds to a machine relation or is trivial; the trivial a -cells are labeled by letters from Y_r .

Similarly, each cell from the left of \mathcal{C} is a trivial a -cell corresponding to a letter from Y_l or it is an α -cell. \square

Lemma 4.5. *Under the assumptions of Lemma 4.4 (2), $\text{space}_H(w_0, w_t) \leq c_3 \max(|w_0|, |w_t|) + c_4$, where the constants c_3, c_4 do not depend on the derivation D .*

Proof. It follows from Lemma 4.4 that all the cells of Δ correspond to the machine relations (3.4), and the derivation D is a peeled machine derivation, where the letter ω is not involved in the commands of the corresponding computation \mathcal{C} . Also there is a reduced computation $\mathcal{C}' : w_0 \rightarrow \dots \rightarrow w_{t'} = w_t$. Then t' is bounded by a linear function of $\min(|w_0|, |w_t|)$ by Lemma 2.10 (e). Since the set of defining relations of H is finite, this implies that the space of \mathcal{C}' , is bounded by a linear function of $\max(|w_0|, |w_t|)$. \square

Lemma 4.6. *Assume that Δ has one through q -band \mathcal{C} , has a through ω -band \mathcal{D} , and \mathcal{C} has no cells having an edge labeled by p . Then*

$$\text{space}_H(w_0, w_t) \leq \max(c_3|w_0| + c_4, c_3|w_t| + c_4, S'_5(\max(|w_0|_a, |w_t|_a)))$$

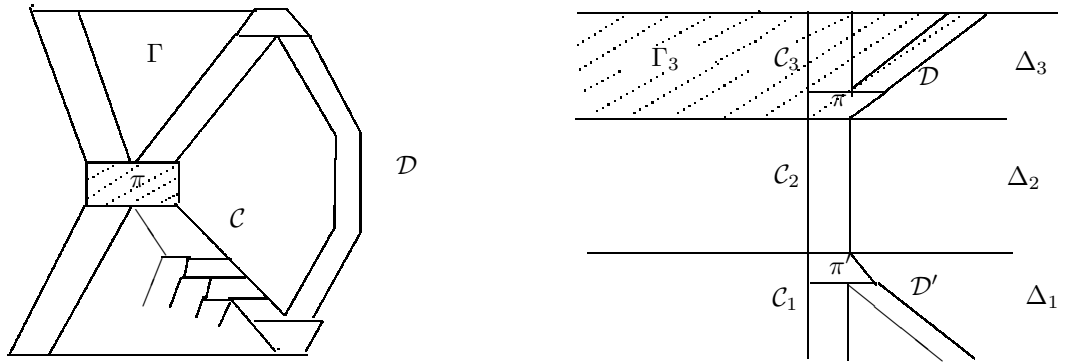
Proof. By Lemma 4.1 a through α -band \mathcal{B} (if any exists) consists of the left-most cells of Δ , and by Lemma 3.16 (d), there are no lenses between \mathcal{C} and \mathcal{D} . Therefore, as in the proof of Lemma 4.5, we obtain that Δ is a machine or a peeled machine trapezium (depending on the presence of a through α -band in it). If it is peeled machine, then $\text{space}_H(w_0, w_t) \leq c_3 \max(|w_0|, |w_t|) + c_4$, as in Lemma 4.5. If Δ is a machine trapezium, then the statement of the lemma follows from the definition of the function $S'_5(n)$. \square

Lemma 4.7. *Assume that Δ has one through q -band \mathcal{C} , and \mathcal{C} has an edge labeled by p . Then $\text{space}_H(w_0, w_t) \leq S'_5(\max(|w_0|, |w_t|) + 5$.*

Proof. By Lemma 4.4 (1), each cell from the left of \mathcal{C} is a trivial a -cell corresponding to a letter from Y_l or it is an α -cell.

Assume that an ω -band \mathcal{D} starts and ends on \mathcal{C} (from the right). Let Γ be the subtrapezium of Δ bounded from the right by the right side \mathbf{z} of \mathcal{D} and bounded from the left by a part of the left side of Δ . By Lemma 3.11 (b), Γ is a (peeled) augmented machine trapezium corresponding to a computation of M_5 . Applying the type separation trick, we may assume that all cells from the right of \mathbf{z} are trivial, and the computation of M_5 is reduced. Moreover, it is non-empty since otherwise the type of \mathcal{C} could be decreased after removing of two derivation bands containing the $q_1\omega$ -cells of Γ .

If Γ has no cell having both an α -edge and a q -edge, then we have a contradiction with Lemma 2.10 (f). Suppose there is such a cell π , and let us choose it to be the closest one to the bottom of Γ . Then by Lemma 2.10 (f), the part of \mathcal{C} between the bottom of Γ and π has no edges labeled by letters from $Y_l \setminus A_l$. Hence only a -bands corresponding to letters from A_l can start on the bottom of Γ and end on \mathcal{C} from the left. Therefore the a -letters from the bottom label of Γ belong to A_l contrary to Lemma 3.12 (b). So we may assume that no ω -band starts and ends on \mathcal{C} .



There is at most one ω -band \mathcal{D} starting with a cell π of \mathcal{C} and ending on the top of Δ and at most one ω -band \mathcal{D}' starting on the bottom of Δ and ending with a cell π' of \mathcal{C} . (We take into account that the p -cells corresponding to the relation $p = q_1\omega$ and to $q_1\omega = p$ alternate in \mathcal{C} .) Since \mathcal{C} has a p -cell, Δ must consists of the following pieces enumerated from the bottom to the top (some of them may be absent): the

subtrapezium Δ_1 crossed by the the part \mathcal{C}_1 of \mathcal{C} , connecting the bottom of Δ and π' , the subtrapezium Δ_2 crossed by the part \mathcal{C}_2 of \mathcal{C} , containing only p -cells corresponding to the auxiliary relations involving letters from A , and the subtrapezium Δ_3 crossed by the part \mathcal{C}_3 of \mathcal{C} , starting with π and ending on the top of Δ . The band \mathcal{C} cannot share a cell with a through ω -band, and so there are no through ω -bands in Δ since Δ is indivisible.

Let Γ_3 is the part of Δ_3 bounded from the right by the right side \mathcal{D} . By Lemma 3.16 (c), Γ_3 is a (peeled) augmented machine trapezium.

If \mathcal{C}_3 has no α -cells, then every a -band starting on \mathcal{C}_3 from the left, corresponds to the a -letter from A_l by Lemma 2.10 (f). If \mathcal{C}_3 has an α -cell, then one can choose such a cell to be the closest to the bottom of Γ_3 and, as above, obtain a contradiction with Lemma 3.12 (b). Similarly, every a -band starting on \mathcal{C}_1 from the left, corresponds to the a -letter from A_l , and therefore the same property holds for the whole \mathcal{C} .

It follows that there are no a -edges corresponding to the letters of $Y_l \setminus A_l$ from the left of \mathcal{C} since the maximal a -band through such edges would make the trapezium Δ divisible. Moreover, we see that if Δ has an α -band \mathcal{B} , then no ω -band starts/ends on \mathcal{C} . Now we consider two cases.

Case 1: Δ has no through α -bands.

One can apply the time separation trick to Δ_3 and its 'left half' Γ_3 . Therefore one can assume that the lenses from the right of \mathcal{C} (if they exist) lie in Δ_2 . Furthermore, if Δ_2 corresponds to a subderivation $w_i \rightarrow \dots \rightarrow w_j$ of D , then $|w_0| = |w_1| = \dots = |w_i|$ and $|w_j| = \dots = |w_t|$ by Lemma 2.10 (f). Hence it suffices to estimate $space_H(w_i, w_j)$.

We have $w_i = UpV$, where U is a word in A_l and V is a word in A . Indeed, every maximal a -band \mathcal{A} of Δ_2 disposed from the right of \mathcal{C} and corresponding to a non- A a -letter cannot end either on the p -cell of \mathcal{C} or on an outer boundary of a thick lens of Δ_2 . So both sides of it divide Δ_2 , and the sides of the maximal extension of \mathcal{A} in the whole Δ divide Δ , a contradiction. Similar form has $w_j = U'pV'$. Hence $\psi(w_i) = \psi(w_j)$ by Lemma 3.15.

To complete the proof, we first use the relations $a_l p = pa$ to replace w_i by $w'_i = pU_A V$, where U_A is the copy of U in the alphabet A . The derivation $w_i \rightarrow \dots \rightarrow w'_i$ has space $|w_i| = |w'_i|$. Similarly we obtain w'_j . By Remark 3.2, $space_H(U_A V, U'_A V') \leq S'_5(\max(|U_A V|, |U'_A V'|)) + 3$. Hence

$$space_H(w_i, w_j) = space_H(w'_i, w'_j) \leq S'_5(\max(|w_0|, |w_t|)) + 4$$

Case 2: Δ has a through α -band \mathcal{B} (as in Lemma 4.1). Since there are no $q_1\omega$ -cells in \mathcal{C} , every cell of this band is a cell corresponding to a relation $pa = a_l p$ or a trivial p -cell. In particular, \mathcal{C} does not share cells with ω -bands, and therefore by Lemma 4.1, Δ has no through ω -bands since both sides of such a band would be vertical, but Δ is indivisible.

Since every a -band starting/ending on \mathcal{C} from the right or on the outer boundary of a thick lens, correspond to a letter from A , the top and bottom labels of Δ from the right of \mathcal{C} are the words in the alphabet A (again, because Δ indivisible). So $w_0 = \alpha UpV$, $w_t = \alpha U'pV'$, where V, V' are words in A (and U, U' are words in A_l).

Hence, by Lemma 3.15, $\psi(w_0) =_S \psi(w_t)$. The relations $pa = a_l p$ preserve the value of ψ and does not change the length. So we may assume that $w_0 = \alpha pV$, $w_t = \alpha pV'$, where V and V' are words in A . By Remark 3.2, the words V and V' can be connected by an H -derivation of space at most $S'_5(\max(|V|, |V'|)) + 3$. Therefore $space_H(w_0, w_t) \leq S'_5(\max(|w_0|_a, |w_t|_a)) + 5$, as required. \square

Summarizing, we obtain

Lemma 4.8. *Assume that Δ is an indivisible minimal trapezium without caps, cups, and A -triangles, and Δ corresponds to a derivation $D : w_0 \rightarrow \dots \rightarrow w_t$ over H . Then*

$$space_H(w_0, w_t) \leq \max(c_3|w_0| + c_4, c_3|w_t| + c_4, S'_5(\max(|w_0|_a, |w_t|_a)) + 5)$$

Proof. By Lemmas 4.2 and 4.3, we may assume that Δ has exactly one through q -band \mathcal{C} . If \mathcal{C} has no p -cells, then the statement of the lemma follows from Lemmas 4.5 and 4.6. Otherwise it follows from Lemma 4.7. \square

Now we want to eliminate the restrictions of Lemma 4.8 imposed on Δ .

Lemma 4.9. *The space function of H is bounded from above by a function equivalent to the function $S'_5(n)$.*

Proof. We define the function $f(n) = S'_5(n + c) + c_3n + c_4 + 5$ for $n \geq 1$, where c is the constant from Lemma 2.10 (d), c_3, c_4 are from Lemma 4.5 (and 4.8), and define $f(0) = 0 (= S'_5(0))$. Obviously, $f(n) \sim S'_5(n)$. We can use the inequality $f(n - k) + k \leq f(n)$ for $0 \leq k \leq n$. Indeed the function $S'(n)$ is non-descending, and one can select $c_3 \geq 1$.

Now we modify the length of a word w : by definition $\|w\|$ is the number of letters, where every α - or q -letter is counted with weight c , and other letters are counted with weight 1.

Note that $|w| \leq \|w\| \leq c|w|$ for every word in the generators of H . Therefore to prove the lemma, it suffices to prove the inequality $\text{space}_H(w, w') \leq f(\|w\| + \|w'\|)$ for any pair of equal in H words w and w' . This will be proved by induction on $\Sigma = \|w\| + \|w'\|$ with trivial base $\Sigma = 0$. So we will assume that $\Sigma > 0$ and consider a derivation $D : w = w_0 \rightarrow \dots \rightarrow w_t = w'$. Let us denote by Δ the corresponding minimal trapezium. Of course, one may assume that the unique vertical line connecting the endpoints of the left side of Δ is the left side itself since otherwise one can replace Δ by a subtrapezium. Similar assumption is taken for the right side of Δ .

First assume that the trapezium Δ is divisible and use the notation of Remark 3.3. Then (see formula (3.6)) there is a derivation

$$D' : w = w_0(1)w_0(2) \rightarrow \dots \rightarrow w_t(1)w_0(2) \rightarrow \dots \rightarrow w_t(1)w_t(2) = w',$$

where $\max(\|w_0(1)\| + \|w_t(1)\|, \|w_0(2)\| + \|w_t(2)\|) < \|w\| + \|w'\|$, and so by the inductive hypothesis, the first half (the second half) of the derivation D' can be chosen with space at most $f(\|w_0(1)\| + \|w_t(1)\|) + |w_0(2)|$ (resp., at most $f(\|w_0(2)\| + \|w_t(2)\|) + |w_t(1)|$.) Hence

$$\begin{aligned} \text{space}(D') &\leq \max(f(\|w\| + \|w'\| - \|w_0(2)\|) + \|w_0(2)\|, f(\|w\| + \|w'\| - \|w_t(1)\|) + \|w_t(1)\|) \\ &\leq f(\|w\| + \|w'\|) \end{aligned}$$

Thus we may further assume that the derivation trapezium Δ is indivisible.

Now assume that there is an A -triangle in Δ . It means that the bottom (or the top) label of Δ is of the form $w = \bar{w}u\bar{\bar{w}}$, where u is a non-empty word in the alphabet A and $u =_S 1$ by Lemma 3.14. By Remark 3.2, $\text{space}_H(u, 1) \leq S'_5(|u|) + 3$. Therefore there is a derivation $w \rightarrow \dots \rightarrow \bar{w}\bar{\bar{w}}$ over H of space at most

$$S'_5(|u|) + 3 + |\bar{w}| + |\bar{\bar{w}}| \leq f(|u|) + |\bar{w}| + |\bar{\bar{w}}| = f(|w| - |\bar{w}| - |\bar{\bar{w}}|) + |\bar{w}| + |\bar{\bar{w}}| \leq f(|w|) \leq f(\|w\|)$$

By the inductive hypothesis, there is a derivation $\bar{w}\bar{\bar{w}} \rightarrow \dots \rightarrow w'$ of space $\leq f(\|\bar{w}\bar{\bar{w}}\| + \|w'\|) \leq f(\|w\| + \|w'\|)$, hence $\text{space}_H(w, w') \leq f(\|w\| + \|w'\|)$. Thus we may further assume that Δ has no A -triangles.

Assume that Δ has a cup (or cap). Let Γ be a cup of minimal height. By Lemma 3.16 (a), there are no lenses enclosed in Γ . Therefore the type of the maximal q -band C_Γ of Γ is 0 or 1 by Lemma 3.12 (c).

If the type of C is 0, then by Lemma 3.17, the word w' is of the form $\bar{w}\alpha W p \bar{\bar{w}}$, where W is a word in A_l . Hence there is a derivation $w' \rightarrow \dots \rightarrow \bar{w}\alpha p W' \bar{\bar{w}} \rightarrow \bar{w}W' \bar{\bar{w}}$, where W' is the copy of W in the alphabet A , and this derivation has space $|w'| \leq \|w'\|$. Since $\|\bar{w}W' \bar{\bar{w}}\| = \|w'\| - 2c$, we obtain by the inductive hypothesis, that

$$\text{space}_H(w, \bar{w}W' \bar{\bar{w}}) \leq f(\|w\| + \|w'\| - 2c) \leq f(\|w\| + \|w'\|)$$

Hence $\text{space}_H(w, w') \leq f(\|w\| + \|w'\|)$, as desired.

If the type of C is 1, then, by Lemma 3.17, the word $B = b(\Gamma)$ is reachable by the machine M_5 . By Lemma 2.10 (d), there is a computation $B \rightarrow \dots \rightarrow B'$ of M_5 , where B' is an input configuration of M_5 and $|B'|_a \leq |B|_a + c$, and so $\|B'\| \leq \|B\| + c$. Denote by C' the corresponding machine derivation over H . Furthermore applying the auxiliary relations in the standard way, one can extend C' , remove the letters α, q_1 and ω from B' and obtain a word B'' with $\|B''\| < \|B'\| - 2c < \|B\| - c$. Hence we have a derivation $w' = \bar{w}B\bar{\bar{w}} \rightarrow \dots \rightarrow \bar{w}B''\bar{\bar{w}}$ of space at most $S'_5(\|B\| + c) + |w'| - |B|$. By the inductive hypothesis, there is a derivation $w \rightarrow \dots \rightarrow \bar{w}B''\bar{\bar{w}}$ of space at most $f(\|w\| + \|w'\| - c)$. Therefore $\text{space}_H(w, w') \leq f(\|w\| + \|w'\|)$, as required.

Thus we may assume that Δ satisfies the assumptions of Lemma 4.8, and therefore $\text{space}_H(w, w') \leq f(\|w\| + \|w'\|)$ by Lemma 4.8 and the definition of $f(n)$. The Lemma is proved.

4.2 Lower bounds and completion of proofs

We define a monoid H' as follows. The set of generators of H' is $A_{H'} = Y_l \sqcup Y_r \sqcup Q \sqcup \{\alpha, \omega\}$, i.e., $A_{H'} = A_H \setminus \{A \cup \{p\}\}$. The set of defining relations of H' consists of only machine relations of H : $R_{H'} = \{V' = V \text{ for every command } V \rightarrow V' \text{ of } M_5\}$.

Lemma 4.10. *The space function of H' is equivalent to $S'_5(n)$.*

Proof. The space function of H' is bounded from above by a function equivalent to $S'_5(n)$. This statement is a very easy version of Lemma 4.9, since the analogs of Lemmas 4.1 - 4.6 and 4.8 become trivial when we have no letters from A , no p , no defining relations of H' with 1 in the left/right sides, and consequently, no triangles, lenses, caps and cups.

By the definition of the set of relations $R_{H'}$, every computation of M_5 can be considered as a derivation over H' , and vice versa, every derivation $w \rightarrow \dots$, where w is a configuration of M_5 , is a computation. Let a derivation C over H' connects $w = \alpha U q V \omega$ and $\alpha U' q' V' \omega$, where $q, q' \in Q$, the words U, U' are words in the alphabet Y_l and V, V' are words in Y_r . Then C is a machine derivation of M_5 since all defining relations of H' are machine relations. Hence if for two configurations w and w' of M_5 , we have $\text{space}_{M_5}(w, w') = s$ for some s , then w and w' cannot be connected by a derivation over H' with space $\leq s$. Hence the space function of H' is at least $S'_5(n)$, and the statement of the lemma follows. \square

Proof of Theorem 1.1. Let a monoid H'' be a copy of H' given by a finite presentation with a set of generators $A_{H''}$ disjoint with A_H . We define the monoid P announced in Theorem 1.1, as the free product $H \star H''$ and consider its space function $s(n)$ with respect to the presentation $\langle A_H \cup A_{H''} \mid R_H \cup R_{H''} \rangle$.

On the one hand, any derivation over P projects on a derivation C over H'' . (One just deletes the letters from A_H in any word from C .) Therefore the space function $s(n)$ of P is greater than or equal to the space function of H'' . Hence $s(n) \geq S'_5(n)$ by Lemma 4.10. On the other hand two words w and w' over $A_H \cup A_{H''}$ are equal in P iff their corresponding A_H - and $A_{H''}$ -syllables are equal in H and in H' , respectively. Since the derivations between equal words can be define syllable-by-syllable, we see that, up to equivalence, $s(n)$ does not exceed the maximum of the space functions of H and of H' . Therefore $s(n) \leq S'_5(n)$ by Lemmas 4.9 and 4.10.

Our estimates show that $s(n) \sim S'_5(n)$. Recall that $S'_5(n) \sim S_0(n)$ by Lemma 2.10 (b). Hence $s(n) \sim S_0(n)$, and by Lemma 3.14, the theorem is proved. \square

Proof of Corollary 1.3. If the function $S_0(n)$ is bounded by a polynomial, then so is $S'_5(n)$ by Lemma 2.10 (b). By Lemma 3.14, S is a subsemigroup (submonoid) of the monoid H , and the space function of H is polynomial by Lemma 4.9.

Conversely, assume that a finitely generated semigroup (monoid) S is embedded in a finitely presented semigroup (monoid) H with polynomial space function $s(n)$. Then the word problem in S is solvable by an NTM with space $\leq s(n)$: This machine takes any word w in the generators of S , rewrites it in the generators of H , and if $w =_H 1$, it produces a derivation $w \rightarrow \dots \rightarrow 1$. (Recall that an NTM may guess and verify. The head of this NTM can move along a word and can replace a subword u by v if $u = v$ or $v = u$ is one of defining relations of H . See details in [2].) By the remarkable theorem of Savitch (see [9], Corollary 1.31), if an NTM has polynomial space complexity, then there exists a DTM solving the same algorithmic problem with a polynomial space as well. Therefore the corollary is proved. \square

Proof of Corollary 1.4. The word problem in a 1-element group S is linear space decidable. But one can force to solve this problem with space complexity $f(n)$ of given deterministic machine M . For this goal, the machine M_0 from Subsection 2.2 should do the following extra work. Given an input word uw' of length n , then in the beginning, M_0 let machine M to use extra tapes and to accept or to reject in consecutive order all words w of length $\leq n$ in the tape alphabet of M . Clearly the space function of such a machine M_0 will be equivalent to $f(n)$. Then we apply Theorem 1.1 to complete the proof. \square

Proof of Corollary 1.5. There exists a finitely presented semigroup S (even a group, see [27] or [28]) with polynomial space (PSPACE) complete word problem (see [9] for the definition). By Theorem 1.1, S is a submonoid of a finitely presented monoid P with polynomial space function, and so the word problem in P is at least PSPACE hard. On the other hand, there is a polynomial $f(n)$ such that two words w and w' are equal in P iff there exists a derivation $w \rightarrow \dots \rightarrow w'$ of space $\leq f(\max(|w|, |w'|))$. It

follows (as in the proof of Corollary 1.3) that there is an NTM of space complexity $\preceq f(n)$ which solves the word problem in P , and so there is a DTM solving the same problem in polynomial time. Thus the corollary is proved. \square

Proof of Corollary 1.6. As we mentioned in Introduction, our notion of space function for semi-groups differs from that used in [22] for groups. Nevertheless the proof of Corollary 1.7 [22] can be literally repeated here to deduce the proof of Corollary 1.6 from Corollary 1.4. \square

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